Reduction by Spinor

Based on the work with Britto,
Anastasiou, Britto, Kunszt, Mastrolia
hep-ph/0609191 and coming soon papers

Plan

- Motivation
- 4D phase space integration
- General dimensional phase integration
- Recursion and Reduction
- Final Remark

(A) Motivation

- One big event for next couple years is the running of LHC.
- With LHC, we can test standard model and search new physics.
- However, all of these depend on our ability to recognize data.
- Data includes the signals as well as backgrounds.

- With the accuracy of data, one loop amplitudes become necessary part for our analysis.
- However, although theoretically we can use Feynman diagrams to do all calculations, practically it is extremely difficult problem.

- The reason for the complexity of amplitudes involving QCD processes are following:
  - Too many Feynman diagrams.
    - 10,525,900 for Tree level 10 gluons.
    - 227,585 for 1-loop 7 gluons.
  - Each diagram has complicated tensor structure.
  - Not very efficient:
    - Huge cancellation between different diagrams to achieve gauge invariance.

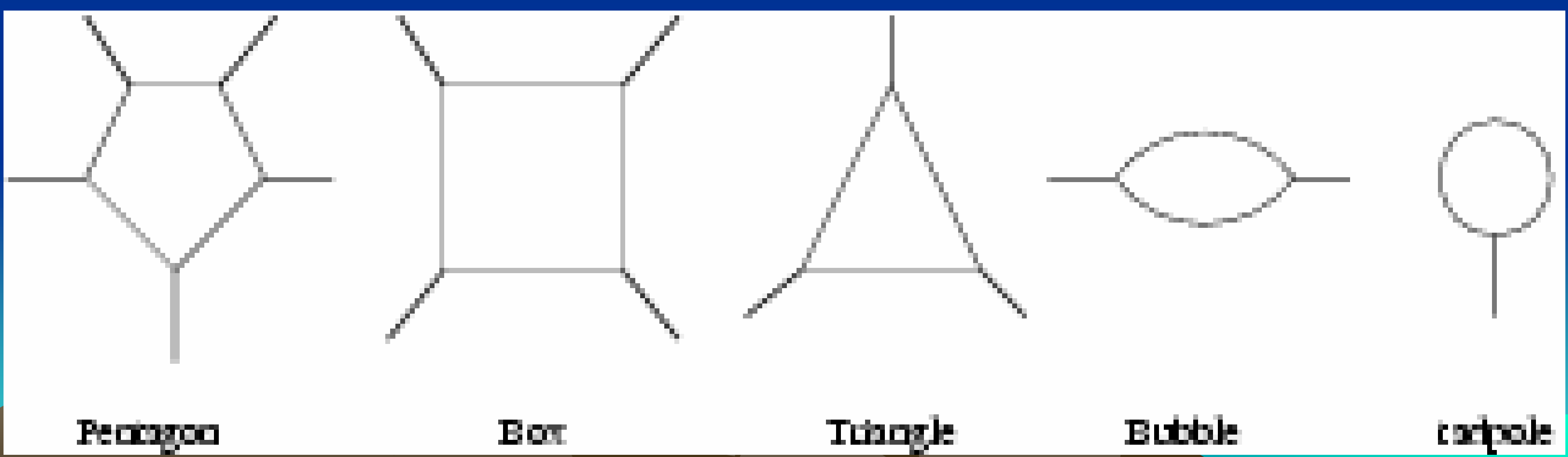
- Need better way to do it!

(A-1) Some Simplifications for Amplitudes:(Dixon, TASI lecture 1996)

- (1) Color decomposition: separate whole amplitudes into minimum gauge invariant pieces and focus on them piece by piece.
- (2) Spinor-helicity formalism: very useful for massless particles because the expression will dramatically simplified.
- (3) Supersymmetry decomposition:
- (4) Expand amplitudes into basis of loop integrals: so the problem is reduced to find coefficients only. This is the standard practice. (Passarino-Veltman reduction).
- (5) Unitary cut method for coefficients of basis: One simple way to read out coefficients by using on-shell tree level amplitudes.

(A-2) Passarino-Veltman reduction:

- The good feature is that it has reduced all integration into a set of master integrals. (Passarino,Veltman)
- The set of basis (master integratals) for one loop amplitudes are pentagons, boxes, triangles, bubbles and tadpoles in general.



- However, in PV program, the calculation of coefficients of these master integrals is not easy:
  - Again many cancellations in middle steps.
  - Unphysical poles in gram determinants.
  - The inverse of large gram matrix.

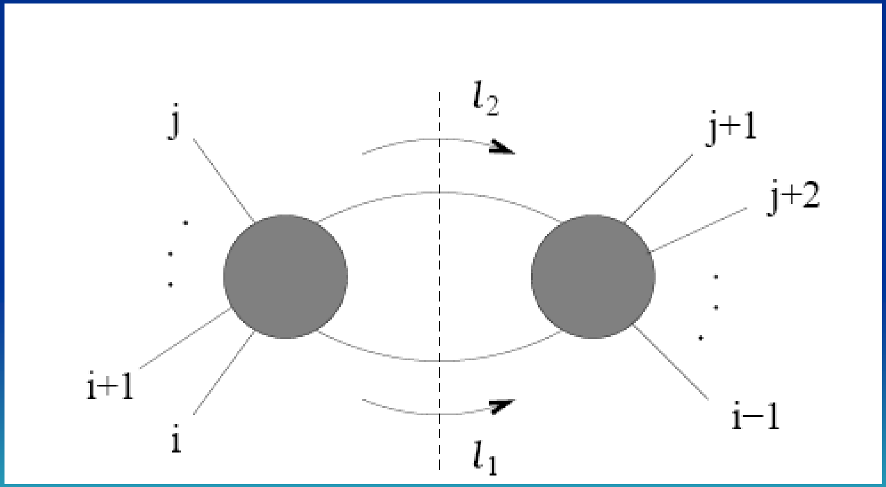
- The need for improvement of PV.

(A-3) Unitary cut method: (Bern, Dixon, Dunbar, Kosower, 1994; Cachazo, 2004; Britto, Cachazo, Feng, 2004.)

- Taking unitary cut at following equation

$$C_{i,\ldots,j} = \Delta A_n^{1-\mathrm{loop}} = \sum B_k \Delta I_k$$

- The left hand side can be calculated by



$$\int d^4\ell \delta(\ell^2)\delta((\ell - K_1)^2) A_L^{tree} A_R^{tree}$$

- ΔI_k is unique for different bases.

- Good points of unitary cut methods:
  - The input is tree-level on-shell amplitudes. Thus gauge invariance and huge cancellations have been taken care of already.
  - Each time, we find subset of coefficients instead of getting all at once.
  - Expression is much more compact.
- The key point becomes how to do phase space integration.

(B) How to Do Cut Integration:

- The main technique is to reduce 2D cut integrations into pure algebraic manipulations, i.e., reading out residue of poles. This amazing technique is first introduced by Cachazo, Svrcek and Witten (Witten; Cachazo, Svrcek, Witten).
- To motivate, let us see one simple example in complex analysis:
  - It is well known

$$\frac{\partial}{\partial \bar{z}} \frac{1}{z-b} = 2\pi \delta^2(z-b)$$

– Using this we can do following integration

$$
\begin{aligned}
& \int dz d\bar{z} \frac{1}{z-b} \partial_{\bar{z}} \frac{1}{\bar{z}-a} \\
= \quad & \int dz d\bar{z} \partial_{\bar{z}} \left(\frac{1}{z-b} \frac{1}{\bar{z}-a} \right) - \int dz d\bar{z} \frac{1}{\bar{z}-a} \partial_{\bar{z}} \frac{1}{z-b} \\
= \quad & - \int dz d\bar{z} \frac{1}{\bar{z}-a} 2\pi \delta^2 (z-b) \\
= \quad & -2\pi \frac{1}{\bar{b}-a}
\end{aligned}
$$

- The key idea is generalize above manipulation in the framework of spinor formalism.

(B-1) Proper measure of cut-integration:

(Cachazo, Svrcek, Witten, 2004)

- Using new parametrization $\ell = t\lambda\widetilde{\lambda}$ measure become

$$\int d^4\ell\delta^+(\ell^2) = \int_0^\infty t dt \int_{\widetilde{\lambda}=\lambda} \langle\lambda\, d\lambda\rangle\, [\widetilde{\lambda}\, d\widetilde{\lambda}]$$

Thus the cut-integration can be written as

$$\int d^4\ell\delta(\ell^2)\delta((\ell-K_1)^2)A_L^{tree}A_R^{tree}$$

$$= \int_0^\infty t dt \int_{\widetilde{\lambda}=\lambda} \langle\lambda\, d\lambda\rangle\, [\widetilde{\lambda}\, d\widetilde{\lambda}]\delta\left(K^2 + t\left\langle\lambda|K|\widetilde{\lambda}\right]\right)$$

$$A_L^{tree}(t,\lambda,\widetilde{\lambda})A_R^{tree}(t,\lambda,\widetilde{\lambda})$$

(B-2) Transform into canonical form: (Britto, Cachazo, BF; Britto, BF, Mastrolia)

- After t-integration, we get sum of terms with following form

$$\frac{1}{\left\langle \lambda | P_{cut} | \widetilde{\lambda} \right]^n} \frac{\prod [a_i\ \widetilde{\lambda}] \prod \langle b_j\ \lambda \rangle \prod \left\langle \lambda | Q_k | \widetilde{\lambda} \right]}{\prod [\widetilde{a}_i\ \widetilde{\lambda}] \prod \left\langle \widetilde{b}_j\ \lambda \right\rangle \prod \left\langle \lambda | \widetilde{Q}_k | \widetilde{\lambda} \right]},$$

- Two important points:
  - (a) The degree of λ and $\widetilde{\lambda}$ is -2;
  - (b) only $\left\langle \lambda | P_{cut} | \widetilde{\lambda} \right]$ has higher pole;

- Next we use Southen identity to split them into canonical form and write canonical form into total derivative as

$$\langle\lambda\, d\lambda\rangle\, [\widetilde{\lambda}\, d\widetilde{\lambda}] G(\lambda,\widetilde{\lambda}) = \langle\lambda\, d\lambda\rangle\, [d\widetilde{\lambda}\, \partial_{\widetilde{\lambda}}] \widetilde{G}(\lambda,\widetilde{\lambda})$$

by using formula

$$[\widetilde{\lambda}\, d\widetilde{\lambda}] \left(\frac{[\eta\, \widetilde{\lambda}]^n}{\left\langle \lambda | P | \widetilde{\lambda} \right]^{n+2}} \right) = [d\widetilde{\lambda}\, \partial_{\widetilde{\lambda}}] \left(\frac{1}{(n+1)} \frac{[\eta\, \widetilde{\lambda}]^{n+1}}{\langle\lambda | P | \eta]\, \left\langle \lambda | P | \widetilde{\lambda} \right]^{n+1}} \right)$$

- Now we reduce to read out residue of poles.

(C) Reduction by Spinor:

- The above cut integration is done in 4D.
- However, because the IR and UV divergent, pure 4D integration will miss some contributions.
- Could we generalize above method to general $(4-2\epsilon)$-dimension?

(C-1) 4D Helicity Scheme:

- The first step is to write measure into following form

$$\int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} = \int \frac{d^4\widetilde{\ell}}{(2\pi)^4} \int \frac{d^{-2\epsilon}\ell_\epsilon}{(2\pi)^{-2\epsilon}} = \int \frac{d^4\widetilde{\ell}}{(2\pi)^4} \frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)} \int d\mu^2 (\mu^2)^{-1-\epsilon}$$

where $(4-2\epsilon)$ momentum p is decomposed as

$$p = \widetilde{\ell} + \vec{\mu} \text{ where } \widetilde{\ell} \text{ is 4-dimensional}$$

- The second step is decomposed massive momentum

$$\widetilde{\ell} = \ell + zK, \qquad \ell^2 = 0,$$

- The measure becomes

$$\int d^4\widetilde{\ell} = \int dz d^4\ell \delta^+(\ell^2)(2\ell \cdot K)$$

- Now we can perform the phase space integration as

$$
\begin{aligned}
C[I_2(K)] &= \int_0^1 du\, u^{-1-\epsilon} \int dz\, d^4\ell\, \delta^+(\ell^2)(2\ell\cdot K)\,\delta(\tilde{\ell}^2 - \mu^2)\,\delta(K^2 - 2K\cdot\tilde{\ell}) \\
&= \int_0^1 du\, u^{-1-\epsilon} \int dz\, d^4\ell\, \delta^+(\ell^2)(2\ell\cdot K)\,\delta(z^2K^2 + 2zK\cdot\ell - \mu^2)\,\delta((1-2z)K^2 - 2K\cdot\ell) \\
&= \int_0^1 du\, u^{-1-\epsilon} \int dz (1-2z)K^2\,\delta(z(1-z)K^2 - \mu^2) \int d^4\ell\, \delta^+(\ell^2)\,\delta((1-2z)K^2 - 2K\cdot\ell)
\end{aligned}
$$

where $u = \frac{4\mu^2}{K^2}$

(D) Reduction and Recursion

- Now we see that we have reduced problem into 4D phase integration plus one-further u-integration.

- But, in fact we do not need to perform u-integration. What we need to do is to find recursion relation.

(D-1) Signature of basis:

$$C[I_2(K)] = \int_0^1 du u^{-1-\epsilon} \sqrt{1-u},$$

$$C[I_3(K_1)] = -\int_0^1 du u^{-1-\epsilon} \frac{1}{\sqrt{\Delta_3}} \ln\left(\frac{Z+\sqrt{1-u}}{Z-\sqrt{1-u}}\right)$$

$$C[I_4(K;P_1,P_2)] = \frac{1}{2K^2} \int_0^1 du u^{-1-\epsilon} \frac{1}{\sqrt{B-Au}} \ln\left(\frac{D-Cu+\sqrt{1-u}\sqrt{B-Au}}{D-Cu-\sqrt{1-u}\sqrt{B-Au}}\right)$$

$$C[I_5(K;P_1,P_2,P_3)] = -\int_0^1 du\, u^{-1-\epsilon}\frac{\sqrt{1-u}}{(K^2)^2} \qquad \text{[Pentagon-gen]}$$

$$\left(\frac{S[Q_3,Q_2,Q_1,K]}{4\sqrt{(Q_3\cdot Q_2)^2 - Q_3^2Q_2^2}} \ln\frac{Q_3\cdot Q_2 - \sqrt{(Q_3\cdot Q_2)^2 - Q_3^2Q_2^2}}{Q_3\cdot Q_2 + \sqrt{(Q_3\cdot Q_2)^2 - Q_3^2Q_2^2}} \right.$$

$$+\frac{S[Q_3,Q_1,Q_2,K]}{4\sqrt{(Q_3\cdot Q_1)^2 - Q_3^2Q_1^2}} \ln\frac{Q_3\cdot Q_1 - \sqrt{(Q_3\cdot Q_1)^2 - Q_3^2Q_1^2}}{Q_3\cdot Q_1 + \sqrt{(Q_3\cdot Q_1)^2 - Q_3^2Q_1^2}}$$

$$\left. +\frac{S[Q_2,Q_1,Q_3,K]}{4\sqrt{(Q_2\cdot Q_1)^2 - Q_2^2Q_1^2}} \ln\frac{Q_2\cdot Q_1 - \sqrt{(Q_2\cdot Q_1)^2 - Q_2^2Q_1^2}}{Q_2\cdot Q_1 + \sqrt{(Q_2\cdot Q_1)^2 - Q_2^2Q_1^2}} \right)$$

(D-2) Recursion of bubbles:

$$C[I_{2m}^{(n)}(K)] \equiv \int_0^1 du\, u^{-1-\epsilon} u^n \sqrt{1-u}$$

$$\begin{aligned} C[I_{2m}^{(n)}(K)] &= -\frac{2}{3}(1-u)^{3/2} u^{-1-\epsilon} u^n \Big|_0^1 + \int_0^1 \frac{2(n-1-\epsilon)}{3}(1-u)^{3/2} u^{-1-\epsilon} u^{n-1} \\ &= \int_0^1 \frac{2(n-1-\epsilon)}{3}\sqrt{1-u}(1-u) u^{-1-\epsilon} u^{n-1} = \frac{2(n-1-\epsilon)}{3}\left(C[I_{2m}^{(n-1)}(K)] - C[I_{2m}^{(n)}(K)]\right) \end{aligned}$$

$$C[I_{2m}^{(n)}(K)] = \frac{(n-1-\epsilon)}{(n+\frac{1}{2}-\epsilon)} C[I_{2m}^{(n-1)}(K)],$$

(D-3) Recursion of Triangles:

$$C[I_{3m}^{(n)}(K_1,K_3)] = \int_0^1 du u^{-1-\epsilon} u^n \ln\left(\frac{Z+\sqrt{1-u}}{Z-\sqrt{1-u}}\right)$$

$$C[I_{3m}^{(n)}(K_1,K_3)] = u^{n-1-\epsilon}\left((Z^2-1+u)\ln\left(\frac{Z+\sqrt{1-u}}{Z-\sqrt{1-u}}\right) - 2Z\sqrt{1-u}\right)\Big|_0^1$$
$$-\int_0^1 du u^{n-2-\epsilon}(n-1-\epsilon)\left((Z^2-1+u)\ln\left(\frac{Z+\sqrt{1-u}}{Z-\sqrt{1-u}}\right) - 2Z\sqrt{1-u}\right)$$

$$C[I_{3m}^{(n)}(K_1,K_3)] = -\frac{(Z^2-1)(n-1-\epsilon)}{(n-\epsilon)}C[I_{3m}^{(n-1)}(K_1,K_3)] + \frac{2Z(n-1-\epsilon)}{(n-\epsilon)}C[I_{2m}^{(n-1)}(K_1)],$$

(D-4) Recursion of box:

$$C[I_{4m}^{(n)}(K;P_1,P_2)] \equiv \int_0^1 du\, u^{-1-\epsilon} \frac{u^n\sqrt{1-u}}{\sqrt{(Q_1\cdot Q_2)^2 - Q_1^2 Q_2^2}} \ln\left(\frac{Q_1\cdot Q_2 - \sqrt{(Q_1\cdot Q_2)^2 - Q_1^2Q_2^2}}{Q_1\cdot Q_2 + \sqrt{(Q_1\cdot Q_2)^2 - Q_1^2Q_2^2}}\right)$$

$$= \int_0^1 du\, u^{-1-\epsilon} \frac{u^n}{\sqrt{B-Au}} \ln\left(\frac{D-Cu-\sqrt{1-u}\sqrt{B-Au}}{D-Cu+\sqrt{1-u}\sqrt{B-Au}}\right) \quad \text{[4m-n]}$$

$$C[I_{4m}^{(n)}(K;P_1,P_2)] = \frac{(n-1-\epsilon)}{(n-\frac{1}{2}-\epsilon)}\frac{B}{A}C[I_{4m}^{(n-1)}(K;P_1,P_2)] - \frac{1}{2(n-\frac{1}{2}-\epsilon)}T \qquad (2.34)$$

$$= \frac{(n-1-\epsilon)}{(n-\frac{1}{2}-\epsilon)}\frac{B}{A}C[I_{4m}^{(n-1)}(K;P_1,P_2)] \quad \text{[Box-rec-2]} \qquad (2.35)$$

$$+\frac{(n-1-\epsilon)C_{Z_1}}{(n-\frac{1}{2}-\epsilon)A\,Z_1}C[I_{3m,Z_1}^{(n-1)}(K_1^{(3)},K_3^{(3)})] + \frac{(n-1-\epsilon)C_{Z_2}}{(n-\frac{1}{2}-\epsilon)A\,Z_2}C[I_{3m,Z_2}^{(n-1)}(K_1^{(3)},K_3^{(3)})]$$

(E) Final remark:

- Now we have a complete PV reduction using spinor formalism.
- It combines the advantage of unitary cut, so it is very efficient and compact.
- It is easy to generalize it to case with massive propagators. (Britto and B.F)
- It is to combine with quadruple cuts and triple cuts to further simplify calculation.

- Applications and further directions:
  - Few 5 particle processes have not been calculated.
  - Almost all 6 particle processes needed for LHC have not be calculated.
  - Possible generalization for higher loops?