Yangians in Gauge Theory and String Theory

(work with L. Dolan and E. Witten)

hep-th/0308089, 0401243, 0411020

Agarwal, Rajeev, Ferretti 0409180, 0508138, 0506095

Erkal's senior thesis 2006

B. Zwiebel to appear

(+ Beisert, Staudacher, Kristjansen, Serban, Tseytlin, Zarembo, Minahan, Plefka. etc..)

Definition of Yangian

Start with the generators J^A of a finite-dimensional symmetry group and their structure constants.

The generators satisfy an algebra:

$$[J^A, J^B] = f_{ABC} J^C$$

The Yangian has additional generators

$$[J^A, Q^B] = f_{ABC} Q^C$$

which generate an infinite-dimensional algebra with certain relations

Serre Relations:

$$[Q^A,[Q^B,J^C]]+[Q^B,[Q^C,J^A]]+[Q^C,[Q^A,J^B]]$$

$$=\frac{1}{24}f^{ADK}f^{BEL}f^{CFM}f_{KLM}\{J^D,J^E,J^F\}$$

$$[[Q^A,Q^B],[J^C,Q^D]]+[[Q^C,Q^D],[J^A,Q^B]]$$

$$=\frac{1}{24}(f^{AGL}f^{BEM}f^{KFN}f_{LMN}f^{CDK}$$

$$+f^{CGL}f^{DEM}f^{KFN}f_{LMN}f^{ABK})\{J^G,J^E,J^F\}$$

Under repeated commutators the Q^As generate an infinite dimensional associative algebra. The first charges are the J^As, the next ones are the Q^As, and the other arise from the commutators of previous generators.

For SU(2) the first Serre relations is trivial.

For SU(N) for N>2, the first Serre relation implies the second one.

In general, you need to check the defining relations to make sure that an Yangian symmetry exists.

Non-linear sigma-model

Let $j^{\mu A}$ be the current operator of a non-linear sigma model in which the target space is, for example, a group manifold.

In addition to being conserved, the Lie algebra valued currents $j_\mu = \sum_A j_\mu^A T^A$ obey

$$\partial_\mu j_\nu - \partial_\nu j_\mu + [j_\mu, j_\nu] = 0$$

As a result, in addition to the usual conserved charges

$$J^A = \int_{-\infty}^{\infty} dx\, j^{0\,A}(x,t)$$

there are extra non-local conserved charges, the first of which are

$$Q^A = f^A_{BC} \int_{-\infty}^{\infty} dx \; \int_{x}^{\infty} dy\, j^{0\,B}(x,t)\, j^{0\,C}(y,t)$$
$$-2 \int_{-\infty}^{\infty} dx\, j^{1\,A}(x,t)$$

These Luscher-Pohlmeyer charges obey the Yangian relations, including the Serre relations.

SU(2) Heisenberg chain

The Hamiltonian is:

$$H = \kappa \sum_{i=1}^{L-1} (J_i^A J_{i+1}^A + \frac{1}{4}) \, ,$$

The Yangian is generated by the usual charges J^A and the bilocal charges

$$Q^A = \epsilon_{ABC} \sum_{i<j} J_i^B J_j^C$$

One can check:

$$[H, Q^A] = \frac{\kappa}{2} ({J^A}_1 - {J^A}_L)$$

As in these examples, Yangians emerge naturally in integrable models

The expansion of the monodromy matrix in terms of the spectral parameter gives the Yangian charges Q^A.
The Yang-Baxter equation encodes the Serre relations

Fadde'ev 9605187, Bernard 9211133, MacKay 0409183

Integrability for strings in ADS

For large $\lambda = g^2 N$, superstrings in AdS can be viewed as a non-linear sigma model where the field takes values in a coset superspace. (Metsaev and Tseytlin 9805028)

$$\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}$$

Bena, Polchinski and Roiban 0305116
found an infinite set of non-local classically conserved charges of the Yangian type.

Integrability on the gauge theory side

(small g^2N)

Minahan-Zarembo 0212208:

The one-loop dilatation operator of the S0(6) subsector of the N=4 SYM is the Hamiltonian of a Heisenberg spin chain, which turned out to be exactly solvable by the method of the algebraic Bethe ansatz.

This result was generalized to the full theory by Beisert 0307015: the full one-loop operator was shown to be a PSU(2,2|4) invariant supersymmetric spin chain, and its integrability was also established.

What about Yangians in N=4 SYM?

We have learned that there is a Yangian symmetry at large g^2N (Bena et al)

We have also learned that there is an integrable structure at small g^2N . It is reasonable to expect a Yangian symmetry there as well.

If we find the *same* integrable structure at both weak and strong coupling, that is good evidence that the theory is integrable for all coupling.

Tree-level Yangian

We assume that at $g^2N = 0$ the Yangian has the same structure as in the spin chain systems:

$$Q^A = f^A_{BC} \sum_{j<k} J^B_j J^C_k$$

Here, the J's are the generators of the superconformal group PSU(2,2|4).

We then show that the Q's satisfy the defining relations of a Yangian, including Serre relations.

Higher loop Yangian

Then we consider loop corrections,
which will have to have the general form

$$\tilde{J}^A = J^A + g^2 N J_2^A + \ldots$$

$$\tilde{Q}^A = Q^A + g^2 N Q_2^A + \ldots$$

preserving the Yangian relations, such as

$$[\tilde{J}^A, \tilde{Q}^B] = f_{ABC} \tilde{Q}^C$$

A first check

To the first order, the commutation relation for the dilatation generator requires that the one-loop dilatation operator (the spin chain Hamiltonian) commutes with the tree level Yangian. This result is heavily based on non-trivial properties of the loop corrections to the dilatation operator. We start with the dilatation generator whose structure constant are the bare conformal dimensions.

At tree level $[D, Q^B] = \lambda^B Q^B$

At one-loop $[\tilde{J}^A, \tilde{Q}^B] = f_{ABC}\tilde{Q}^C$ becomes

$$[\delta D, Q^B] + [D, \delta Q^B] = \lambda^B \delta Q^B$$

But

$$[D, \delta Q^B] = \lambda^B \delta Q^B$$

since loop corrections don't mix operators that have different dimensions in the classical limit

So

$$[\delta D, Q^B] = 0$$

The techniques used rely on the special decomposition of the tensor product of the Hilbert spaces of one-particle states into the direct sum of irriducible representations of PSU(2,2|4), subspaces with given lowest conformal dimension.

$$V_i \otimes V_{i+1} = \oplus_{j=0}^{\infty} W_j$$

The one-loop correction to the dilatation operator can be written

$$\delta D = H = \sum_{i=1}^{L} \sum_{j=0}^{\infty} 2h(j) P_{i,i+i}(j)$$

Then we prove

$$[H, Q^A] = J_1{}^A - J_L{}^A$$

Higher-Loop Yangians

We would like to compute the loop corrections to the tree-level Yangian. This higher loop Yangian must satisfy all the defining relations. Also, one can show as before that the complete loop dilatation operator must commute with the exact Yangian, a useful criterion to determine loop corrections to the Yangian.

$$[\tilde{D}, \tilde{Q}^B] = \lambda^B \tilde{Q}^B$$

$$[D, \tilde{Q}^B] = \lambda^B \tilde{Q}^B$$

$$[\Delta D, \tilde{Q}^B] = 0$$

Many perturbative corrections to the PSU(2,2|4) generators have been computed.

As one goes up in perturbation theory, the range of interaction of the spin chain increases. For example, the three-loop dilatation operator can be imbedded in a particularly long-range spin chain, known as the Inozemtsev spin chain.

Hence, the determination of the loop corrections to the Yangian is not simple.

Strategy

Use the corrections to PSU(2,2|4) generators.

$\tilde{Q}^A$ is bilinear in the exact $\tilde{J}^A$s, plus a “local” correction that must be determined.

Make a general ansatz for this correction, with unknown coefficients.

Work in a consistent subsector of the theory

Determine the coefficients by using a relation that you know the Yangian must satisfy at that order.

Check the defining relations, including Serre’s.

Complicated computer codes required

Some New Results

Agarwal and Rajeev computed the 1-loop correction to the SU(2) Yangian

Erkal extended this calculation to 4 loops

Agarwal computed the 1-loop correction for the SU(1|1) sector

Zwiebel extended it to the SU(2|1) sector

All Yangian relations can be proved up to boundary terms, hence they are exact only for infinite chains

Conclusions

- Integrability occupies an increasingly important role in direct test of ADS/CFT correspondence. Integrable structures have emerged on both sides of the correspondence and survive at higher loops.
- Yangian structure appears on both sides.
- All indications say it holds at higher loops.

This is compatible with Beisert et al result on the