Classification of Supersymmetric Backgrounds of String Theory

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Main results based on:

- Systematics of M-theory spinorial geometry
  U. Gran, G. Papadopoulos, D.R.; hep-th/0503046,
- Systematics of IIB spinorial geometry
  U. Gran, J. Gutowski, G. Papadopoulos, D.R.; hep-th/0507143,
- Maximally supersymmetric G-backgrounds of IIB supergravity
  U. Gran, J. Gutowski, G. Papadopoulos, D.R.; hep-th/0604079,
- $N = 31$ is not IIB
  U. Gran, J. Gutowski, G. Papadopoulos, D.R.; hep-th/0606049.

Introduction

Supersymmetric solutions of supergravity play an important role in string/M-theory:

- entropy matching: black holes and configurations of M-/D-branes,
- phenomenology: flux compactifications to $\mathcal{N} = 1$ in four dimensions,
- AdS/CFT: $AdS_5 \times S^5$, its Penrose limit and deformations with less supersymmetry,
- .....

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Usually via Ansätze of metric and fluxes based on physical intuition. Possible to miss unexpected and exciting backgrounds (e.g. black rings).

Desirable to have classifications!

Classifications

What about classifications? e.g.

- $D = 4$: minimal $\mathcal{N} = 2$ [a] and coupled to vectors [b]
- $D = 5$: minimal $\mathcal{N} = 1$ [c]
- $D = 6$: minimal $\mathcal{N} = 1$ [d]

Theories with 8 supersymmetries and solutions with $N = 4, 8$.

Two techniques: Newman-Penrose ('82, '83) & spinor bilinears ('02,...).

[a]: Gibbons, Hull '82, Tod, '83, [b]: Meessen, Ortín '06, [c]: Gauntlett et al '02, [d]: Gutowski, Martelli, Reall '03.

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We propose spinorial geometry as a new technique for more complicated cases such as maximal supergravity.

We will focus on IIB since that is the most difficult case (coset scalars, more - complex & self-dual - fluxes, chiral fermions), but the story generalises to other supergravities.

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Supersymmetric backgrounds with fluxes

Without fluxes: susy $\Rightarrow$ special holonomy - Berger classification.

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With fluxes: susy $\Rightarrow$ G-structures - no classification.

Supercovariant connection $\neq$ Levi-Civita connection:

$$D = d + \Omega + F \neq \nabla = d + \Omega$$

Holonomy(D) $\neq$ gauge group $Spin(dim-1,1)$

Massless/ungauged maximal supergravities: holonomy $SL(32,\mathbb{R})$ [a]

11D: $D_m = \partial_M + \frac{1}{4}\Omega_{M,PQ}\Gamma^{PQ} - \frac{1}{288}(\Gamma_M{}^{PQRS}\mathrm{F}_{PQRS} - 8\mathrm{F}_{MPQR}\Gamma^{PQR})$

$\Rightarrow Spin(10,1)$ $\Rightarrow$ extends this to $SL(32,\mathbb{R})$

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Inequality is reason for complications with fluxes
(e.g. zero curvature $\not\Rightarrow$ trivialisable connection
$\not\Rightarrow$ gauge where Killing spinors are constant).

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Solving the Killing spinor equations

Which G-structures does one find? Different methods to solve the KSE:

Spinor bilinears:

- Uses the N^2 relations $\nabla\kappa_{ij} \sim F\kappa_{ij}$ where $\kappa_{ij} = \bar{\epsilon}_i \Gamma^{(p)} \epsilon_j$,
- Necessary conditions, check sufficiency by hand,

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Spinorial geometry:

- Basis in the space of spinors and description in terms of forms,
- Analyses the N Killing spinor equations $D\epsilon_i = 0$ directly,
- Necessary and sufficient conditions.

Basis in space of spinors

Spinor in terms of forms [a]:

space of Dirac spinors ϵ of $Spin(9,1)$ dimension 64	$\equiv$	space of forms η spanned by $e_1, \ldots, e_5$ (with compl. coeff.) dimension $2 \cdot 2^5$

Weyl spinors $\equiv$ even/odd forms & Majorana spinors $\equiv \eta^* = \Gamma_{6789}\eta$.

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Γ_M-matrices in null and holomorphic basis $M = (-,+,\alpha,\bar{\alpha})$:

$$\begin{array}{lll} \Gamma_a\eta = \sqrt{2}e_a \wedge \eta & \text{for } a = (-,\alpha) & \text{creation operators} \\ \Gamma_{\bar{a}}\eta = \sqrt{2}e_a \lrcorner \eta & \text{for } \bar{a} = (+,\bar{\alpha}) & \text{annihilation operators} \end{array} \quad \Leftrightarrow$$

$SU(4)$-covariant action of Γ-matrices on spinor.

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IIB Killing spinor equtions

Arbitrary IIB spinor: (with $a = (\alpha, 5)$)

$$\epsilon = \underbrace{(f_1 + if_2)1}_{2\cdot 1} + \underbrace{(g_1^{a_1a_2} + ig_2^{a_1a_2})e_{a_1a_2}}_{2\cdot 10} + \underbrace{(h_1^{a_1\cdots a_4} + ih_2^{a_1\cdots a_4})e_{a_1\cdots a_4}}_{2\cdot 5}$$

(f_i, g_i, h_i functions of space-time coordinates)

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(f_i, g_i, h_i functions of space-time coordinates)

Substitute ϵ into IIB Killing spinor eqs:

$$D_M\epsilon = (\partial_M + \tfrac{1}{4}\Omega_{M,N_1N_2}\Gamma^{N_1N_2} - \tfrac{i}{2}Q_M + \tfrac{i}{48}\Gamma^{N_1\ldots N_4}F_{MN_1\ldots N_4})\epsilon$$
$$-\tfrac{1}{96}({\Gamma_M}^{N_1N_2N_3}G_{N_1N_2N_3} - 9\Gamma^{N_1N_2}G_{MN_1N_2})(C\epsilon)^* = 0\,,$$
$$A\epsilon = P_N\Gamma^N(C\epsilon)^* + \tfrac{1}{24}G_{N_1N_2N_3}\Gamma^{N_1N_2N_3}\epsilon = 0\ ,$$

and expand in basis (amounts to products of Γ-matrices), and set all coefficients equal to zero [a].

KSE reduces to linear system of equations for scalars, fluxes, spin connection and functions f (and their derivatives) of Killing spinor.

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Problem of classifying supersymmetric solutions is reduced to parametrising the N Killing spinors and solving the linear system.

[a]: Papadopoulos et al ’04, ’05, Mac Conamhna ’04, ’05.

Examples

- $N = 1$: Use Lorentz symmetry to bring Killing spinor to one of the three orbit representatives with stability subgroup G:

  $G = Spin(7) \ltimes \mathbb{R}^8$ with $\epsilon = (f_1 + if_2)(1 + e_{1234})$

  $G = SU(4) \ltimes \mathbb{R}^8$ with $\epsilon = (f_1 + if_2)1 + (h_1 + if_2)e_{1234}$

  $G = G_2$ with $\epsilon = f_1(1 + e_{1234}) + ig_1(e_{15} + e_{2345})$

  and plug into KSE [a].

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  and plug into KSE [a],
- Maximal number of G-invariant spinors: simple embedding of G [b],
- $N = 31$: Use Lorentz symmetry to bring orthogonal spinor to one of the three orbit representatives [c].

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Maximally supersymmetric G-backgrounds

Maximal number of G-invariant Killing spinors [a]:

$G = \ N =$	1	2	3	4	6	8	16	32
G_2	—	—	—	$\odot$	—	—	—	—
$SU(3)$	—	—	—	—	—	$\odot$	—	—
$SU(2)$	—	—	—	—	—	—	$\odot$	—
1	—	—	—	—	—	—	—	$\odot$
$Spin(7) \ltimes \mathbb{R}^8$	—	$\odot$	—	—	—	—	—	—
$SU(4) \ltimes \mathbb{R}^8$	—	—	—	$\odot$	—	—	—	—
$Sp(2) \ltimes \mathbb{R}^8$	—	—	—	—	$\odot$	—	—	—
$(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$	—	—	—	—	—	$\odot$	—	—
$\mathbb{R}^8$	—	—	—	—	—	—	$\odot$	—

Flux deformations of gravitational solutions with same Killing spinors.

[a]: Gran, Gutowski, Papadopoulos, D.R. '06.

Maximally supersymmetric G-backgrounds

Compact $G = G_2, SU(3), SU(2)$: [a]

Direct product:

($AdS_{d/2} \times S^{d/2}$ or Hpp or $\text{Mink}_{1,d-1}$) $\times$ (manifold M_{10-d} of special holonomy G)

Fluxes constrained to d-dimensional part

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Fluxes constrained to d-dimensional part

Non-compact $G = K \ltimes \mathbb{R}^8$, $K = Spin(7), SU(4), Sp(2), SU(2) \times SU(2), 1$ [a]

pp-wave on 8D manifold with special holonomy K

null fluxes P_-, G_{-mn}, F_{-mnpq}

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Alg. KSE of IIB $\Rightarrow$ scalars and three-form field strengths vanish. [a]

No more c.c. in the diff. KSE of IIB $\Rightarrow$ N even $\Rightarrow$ $N = 32$.
NO $N = 31$ SOLUTIONS IN IIB!

First constraint on N in type II theories.

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Same result in IIA using moving G-frame method [b].

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$N = 31$ in M-theory remains an open question.

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Outlook

- Maximal G-backgrounds of IIB $\surd$
- $N = 31$ $\surd$
- Half-maximal G-backgrounds of IIB?
  - $G = SU(3)$ **with** $N = 4$: AdS/CFT and flux compactifications [a]
  - all $N = 16$ solutions (only one case remaining: half-max. 1)
- ... etcetera! Classification of all supersymmetric solutions?
- T-duality?
- relation to Hitchin's generalised geometry? [b]

[a]: Graña et al '04, '05, [c]: Hitchin '02, Gualtieri '03, '04.

IIB status report

$G =\ \backslash\ N =$	1	2	3	4	6	8	16	31	32
G_2	$\surd$	$\odot$	...	$\surd$	—	—	—	—	—
$SU(3)$	—	...	...	$\odot$	...	$\surd$	—	—	—
$SU(2)$	—	...	...	...	...	$\odot$	$\surd$	—	—
1	—	...	...	...	...	...	$\odot$	$\surd$	$\surd$
$Spin(7) \ltimes \mathbb{R}^8$	$\surd$	$\surd$	—	—	—	—	—	—	—
$SU(4) \ltimes \mathbb{R}^8$	$\surd$	$\surd$	...	$\surd$	—	—	—	—	—
$Sp(2) \ltimes \mathbb{R}^8$	—	...	$\odot$	...	$\surd$	—	—	—	—
$(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$	—	...	...	$\odot$	...	$\surd$	—	—	—
$\mathbb{R}^8$	—	...	...	...	...	$\odot$	$\surd$	—	—

$\surd$: Killing spinor equations have already been solved,
$\odot$: backgrounds with half-maximal number of G-invariant spinors,
...: have more complicated linear systems,
—: do not occur.