Calibrated cycles and T–duality

in collaboration with
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based on

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see also: Talk by Luca Martucci

Outline

Motivation

- Description of structured submanifolds in Hitchin's generalised geometry. [Hitchin, Gualtieri]
- Obtain calibration conditions for submanifolds of geometries with $G \times G$–structure. [Dadok, Harvey, Lawson]
- Description of branes in this framework. Of particular interest is the type II case with $SU(3) \times SU(3)$–structure. [Koerber, Martucci, Smyth]
- Understand how T–duality acts on the calibration conditions. [Ben-Bassat, Boyarchenko]

G–structures

- Consider a manifold M^n with tangent bundle T. The transition functions lie in $GL(n)$.
- A reduction of the structure group is equivalent to the existance of invariant objects.
- The different choices for a subgroup G are parametrised by the coset $GL(n)/G$.
- Therefore the existance of a reduction to G requires the existance of a section of $GL(n)/G$.

Example

A Riemannian metric is stabilised by $O(n)$, the choice of a metric is equivalent to a reduction from $GL(n)$ to $O(n)$. The choice of an orientation reduces the group to $SO(n)$. However, this is only possible if $w_1(M) = 0$.

Generalised geometry

- Consider the vector bundle $T \oplus T^*$ over a manifold M^n.
- Elements of $T \oplus T^*$ are of the form $x \oplus \xi$.
- We have a natural inner product $\langle x, \xi \rangle = -\frac{1}{2}\xi(x)$ of split signature. This reduces the structure group from $GL(2n)$ to $O(n,n)$.
- The choice of an orientation reduces the structure group to $SO(n,n)$.
- A special element of $SO(n,n)$ are the two–forms b, acting on $x \oplus \xi$ as

$$e^b(x \oplus \xi) = \begin{pmatrix} 1 & 0 \\ b/2 & 1 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} .$$

Spinors

- In general the existance of a spin structure requires that the second Stiefel-Whitney class vanishes, $w_2(M) = 0$.
- On $T \oplus T^*$ we have that

$$w_2(T \oplus T^*) = w_2(T) + w_1(T) \cup w_1(T^*) + w_2(T^*) = 0,$$

  since $w_k(T) = -w_k(T^*)$.
- Therefore $T \oplus T^*$ is always spinnable.
- Elements ρ of $Spin(n,n)$ can be regarded as even or odd forms of mixed degree. We get the irreducible spin representations $S_\pm = \Lambda^{ev,od} T^* \otimes \sqrt{\Lambda^n T^n}$.

Spinors

- The Clifford–action of elements in $T \oplus T^*$ on spinors is defined as $(x \oplus \xi) \bullet \rho = -x \llcorner \rho + \xi \wedge \rho$.
- The tranformation under the action of a two form b is given by $e^b \wedge \rho$.
- We can define a bilinear form on $S_\pm$ as $\langle \rho, \tau \rangle = [\rho \wedge \hat{\tau}]^n$, where the $\hat{}$ is an anti-automorphism defined as $\hat{a^p} = (-1)^{p(p+1)/2}$ and $[\cdot]^n$ denotes a projection on the forms of degree n.
- *Pure spinors* are defined by the property that their annihilator, the space

$$W_\rho = \{x \oplus \xi \in T \oplus T^* | (x \oplus \xi) \bullet \rho = 0\}$$

is of maximal dimension.

$G \times G$ structures

- The existance of a metric g and a two–form b induces a reduction from $SO(n,n)$ to $SO(n) \times SO(n)$. (g,b) is sometimes also called a *generalised metric*.
- For spinors $\rho \in S_\pm$ this allows a description in terms of two chiral spinors $\Psi_L, \Psi_R \in Spin(n)$ as $\rho = e^b[\Psi_L \otimes \Psi_R]$.
- If we are given a pair (Ψ_L, Ψ_R), this reduces the structure group to $G_L \times G_R$.

Example

For $n = 6$ we get a reduction to $SU(3) \times SU(3)$ (as the existance of one spinor reduces to $SU(3)$ in the classical case), for $n = 7$ we get $G_2 \times G_2$, $n = 8$ gives $Spin(7) \times Spin(7)$, etc.

Classical calibrations

- For a Riemannian metric g and a p–form ω^p, the restriction of ω^p to a p–dimensional subspace U with

$$j_U : U \hookrightarrow T$$

induces a volume form.

- This can be compared with the Riemannian volume vol_U. ω defines a *calibration*, if we have that

$$j_U^* \omega^p \leq vol_U$$

for any U.

Generalised calibrations

- A generalised calibration can be defined in terms of an even or odd spinor $\rho \in S_\pm$.
- Instead of considering a p-submanifold U alone, we take the pair (U, F), $F \in \Lambda^2 U^*$.
- The associated form $\rho_{U,F} = e^F \wedge \star\widehat{vol_U}$ is *pure*, seen as a $T \oplus T^*$–spinor.

Definition

A spinor ρ defines a calibration, if $\langle \rho, \rho_{U,F} \rangle \leq 1$.

Generalised calibrations

- To make this look more like the classical case, we can rewrite this to

$$[e^{-F} \wedge j_U^* \rho]^p \leq \sqrt{\det\left(j_U^*(g+b) - F\right)}\, vol_U$$

- If ρ is closed, we have that [Koerber]

$$I = \int_U \sqrt{\det(g_{|U} + b_{|U} - F)} vol_U$$

  is minimised.

- If we have $d_H e^{-\phi}\rho = d_H(e^b \wedge C)$, a submanifold calibrated by ρ minimises [Martucci, Smyth]

$$I = \int_U e^{-\phi} \sqrt{\det(g_{|U} + b_{|U} - F)}\, vol_U - \int_U e^{b_{|U} - F} \wedge C_{|U}.$$

$SU(3) \times SU(3)$

Example

In the case of $n = 6$ we obtain in particular also the $SU(3)$–cases. There the calibration form can be written as $\rho = e^b[\Psi \otimes \Psi]$ and we get for an even calibration form $\rho^e v = e^{i\omega}$ and $p = 2k$ the condition (*B-branes*)

$$\frac{1}{k!}\left(j_U^*\omega + F\right)^k = \sqrt{\det(j_U^* g - F)}\, vol_U \, .$$

Choosing an odd calibration form (*A-branes*), we get

$$\left[e^{-F} \wedge j_U^* \mathrm{Re}\Omega\right]^p = \sqrt{\det(j_U^* g - F)}\, vol_U \, ,$$

which includes Langrangian as well as coisotropic branes. In the generalised case there might also be the possibility to obtain *isotropic* branes [Chiantese].

$G_2 \times G_2$

Example

For $n = 7$ we have a $G_2 \times G_2$–structure. The calibration form can be written as (with $c/s = \cos / \sin \left(\sphericalangle(\Psi_L, \Psi_R)\right)$)

$$
\begin{aligned}
[\Psi_L \otimes \Psi_R]^{ev} \quad &= \quad c + s\omega + c(\alpha \wedge \mathrm{Im}\Omega - \frac{1}{2}\omega^2) \\
&\quad -s\alpha \wedge \mathrm{Re}\Omega + -\frac{1}{6}s\omega^3
\end{aligned}
$$

for a one–form α, a two–form ω and a holomorphic three–form Ω. In particular we get the classical case, in which the calibration condition reads

$$
e^{-F} \wedge j_U^*(1 - \star\varphi) \leq \sqrt{\det(j_U^* g - F)} vol_U
$$

and defines a coassociative four–cycle with anti-self-dual field strength.

T–Duality

T–Duality is expected to do the following

- Exchange even and odd calibration forms (guided by the classical case, where we know that A- and B-branes are exchanged).
- Respect the integrability condition $(d_H e^{-\phi}\rho = d_H(e^b \wedge C))$.

We can describe a T-duality transformation [Bouwknegt, Evslin, Mathai] on $T \oplus T^*$ by a one-form θ and it's vertical vector field X, i.e. $\theta(X) = 1$. Therefore $\mathcal{M}_\theta := X \oplus \theta \in Pin(n,n)$. Conjugation of the pair (g,b) with $\mathcal{M}$ projected onto $O(n,n)$ gives a set (g^T, b^T) as predicted by the usual Busher rules.

T–Duality

- The action of $\mathcal{M}_\theta$ on a spinor ρ is given by $\mathcal{M}_\theta \bullet \rho$.
- This preserves the $Spin(n,n)$ orbit–structure and gives an isomorphism between $\Lambda^{ev,od}$ and $\Lambda^{od,ev}$.
- For bi–spinors $\rho = e^b[\Psi_L \otimes \Psi_R]$ we get that $\mathcal{M} \bullet \rho = (-X\llcorner + \theta\wedge)e^b[\Psi_L \otimes \Psi_R] = \|X\|\, e^{b^T}[\Psi_L^T \otimes \Psi_R^T]$.
- In particular bi–spinors of equal chirality are mapped into spinors of opposite chirality and vice versa (as expected from the exchange of type IIA and IIB theories).

T–Duality and calibrations

Let ρ define a calibration for (g, b). Then the application of $\mathcal{M}_\theta$ on $\langle \rho, \rho_{U,F} \rangle \leq 1$ gives

$$(-1)^{n+1} \langle \rho^T, \mathcal{M}_\theta \bullet \rho_{U,F} \rangle \leq 1.$$

Since $\mathcal{M}_\theta$ is orbit and norm preserving, $(-1)^{n+1} \rho^T_{U,F}$ is pure and of unit norm, so it equals ρ_{U^T, F^T} for some suitable (U^T, F^T).

$\rightsquigarrow$ T-Duality transformations preserve calibrations.

T–Duality and calibrations

If we have $d_H e^{-\phi}\rho = d_H(e^b \wedge C)$ such that the calibrated subspaces (U, F) minimise a functional $I(\phi, C)$, we find that

$$X \llcorner d_H \rho - \theta \wedge d_H \rho \cong d_{H^T} \, \|X\| \, \rho^T ,$$

and the subspaces (U^T, F^T) minimise the functional $I(\phi^T, C^T)$, with

$$\phi^T = \phi - ln \, \|X\|$$

and

$$C^T = e^{-b^T} \wedge (-X \llcorner + \theta \wedge) e^b \wedge C.$$

Conclusions

Summary

- We find a general calibration confition for spaces with $G \times G$–structure.
- The known classical cases for $SU(3)$ or G_2 structure are included, as well as the $SU(3) \times SU(3)$ case.
- Calibration is preserved by T-Duality, which maps calibrated cycles wrt to one geometry into calibrated cycles wrt to the T-dual geometry.

Outlook

- Possible to understand (co-)isotropic branes better in this framework?
- Find explicit examples of new brane configurations for accessible models. [Graña, Minassian, Petrini, Tomasiello]

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