Black Hole Entropy Functions and Attractor Equations

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Static Extremal Black Holes

Consider a theory of gravity coupled to several Abelian gauge fields $F^I_{\mu\nu}$ and neutral scalar fields ϕ in $D = 4$. General coordinate invariance, higher-derivative curvature terms.

Static extremal black hole solutions: near-horizon solution determined by a set of attractor equations. Both the attractor equations and the entropy follow from a variational principle based on an entropy function.

The entropy function can be derived from a reduced action evaluated in the near-horizon geometry. `Sen, hep-th/0506177`

In $D = 4$: near-horizon geometry is $AdS_2 \times S^2$. Take the background fields to respect these symmetries:

$$
\begin{aligned}
ds^2 &= v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 \, d\Omega_2 \\
F^I_{rt} &= e^I \quad , \quad F^I_{\theta\varphi} = p^I \sin\theta \\
& v_1 , \; v_2 , \; e^I , \; \phi = \text{constant}
\end{aligned}
$$

Entropy Function and E & M Duality

Reduced Lagrangian

$$\mathcal{F}(e^I, p^I, v_1, v_2, \phi) = \int_{S^2} d\theta\, d\varphi\, \sqrt{|g|}\, L$$

Lagrangian does not transform as a function under electric-magnetic duality: see `de Wit, hep-th/0103086`

$$\tilde{\mathcal{F}}(\tilde{e}, \tilde{p}, \dots) \neq \mathcal{F}(e, p, \dots)$$

Combinations that transform as a function are

$$\mathcal{F} + \frac{1}{2}\left[q_I\, e^I + p^I\, f_I\right] \quad , \quad q_I\, e^I - p^I\, f_I$$
$$q_I = -\partial\mathcal{F}/\partial e^I \quad , \quad f_I = -\partial\mathcal{F}/\partial p^I$$

Combination that is independent of f_I,

$$\mathcal{E}(q, p, v_1, v_2, \phi) = -q_I\, e^I - \mathcal{F}(e, p, v_1, v_2, \phi)$$

Legendre transform. Sen's entropy function.

Black Hole Potential

Extremizing $\mathcal{E}$ with respect to v_1, v_2 and ϕ yields the attractor equations, whose solution determines the horizon values of v_1, v_2 and ϕ in terms of the charges (q_I, p^I).

The value of $\mathcal{E}$ at extremum is the macroscopic black hole entropy:

$\mathcal{S}_{\rm macro} = \pi\, \mathcal{E}|_{\rm extr} = \mathcal{S}_{\rm macro}(q,p)$ `Sen, hep-th/0506177`

Wald's entropy (reduces to Bek-Hawk area law in the absence of higher-derivative terms).

Two-derivative level: $L = -\frac{i}{4}\left(\mathcal{N}_{IJ}\, F^{+I}_{\mu\nu}\, F^{+\mu\nu J} - \bar{\mathcal{N}}_{IJ}\, F^{-I}_{\mu\nu}\, F^{-\mu\nu J}\right)$

$F^{\pm I}_{\mu\nu}$ (anti)selfdual field strength. $\mathcal{N}_{IJ} = \mathcal{N}_{IJ}(\phi)$

Reduced Lagrangian $\mathcal{F}$ at most quadratic in e^I and p^I $\Longrightarrow$

$$\mathcal{E} \propto \left(q_I - \mathcal{N}_{IL}\, p^L\right)\, \left[({\rm Im}\, \mathcal{N})^{-1}\right]^{IJ}\, \left(q_J - \bar{\mathcal{N}}_{JK}\, p^K\right)$$

Indeed compatible with E $\&$ M duality, and equals the black hole potential of `Ferrara + Gibbons + Kallosh, hep-th/9702103`

Application to $N = 2$ Supergravity Theories

Calabi-Yau compactifications of type II string theory: effective Wilsonian Lagrangian contains $F^I_{\mu\nu}$ as well as higher-curvature interactions $\propto$ Weyl^2.

Couplings encoded in holomorphic homogeneous function $F(Y, \Upsilon)$.

Complex scalar fields Y^I. $\Upsilon \propto T^2_{\mu\nu}$ resides in Weyl multiplet.

Compute the entropy function for this class of theories.

`Sahoo + Sen, hep-th/0603149`

Result can be recast in the following way:

`C + de Wit + Mahapatra, to appear`

define

$$
\begin{aligned}
F_I &= \partial F(Y,\Upsilon)/\partial Y^I \quad , \quad F_\Upsilon = \partial F(Y,\Upsilon)/\partial \Upsilon \quad , \quad N_{IJ} = i\left(\bar{F}_{IJ} - F_{IJ}\right) \\
\mathcal{P}^I &= p^I + i\,(Y^I - \bar{Y}^I) \quad , \quad \mathcal{Q}_I = q_I + i\,(F_I - \bar{F}_I) \\
U &= v_1/v_2
\end{aligned}
$$

Application to $N = 2$ Supergravity Theories

Entropy function can be written as

$$\begin{aligned}\mathcal{E} &= U\left[\Sigma + \left(\mathcal{Q}_I - F_{IJ}\,\mathcal{P}^J\right) N^{IK} \left(\mathcal{Q}_K - \bar{F}_{KL}\,\mathcal{P}^L\right)\right] \\ &\quad + \frac{8i}{\sqrt{-\Upsilon}}\,(1-U)\,\left(\bar{Y}^I\,F_I - Y^I\,\bar{F}_I\right) \\ &\quad - i(F_\Upsilon - \bar{F}_\Upsilon)\left[64\,(U + U^{-1} - 2) - 4\,U\,\Upsilon - 16\,(1+U)\sqrt{-\Upsilon}\right]\end{aligned}$$

where

$$\Sigma = -i\left(\bar{Y}^I\,F_I - Y^I\bar{F}_I\right) - 2i(\Upsilon F_\Upsilon - \bar{\Upsilon}\bar{F}_\Upsilon) + p^I(F_I + \bar{F}_I) - q_I(Y^I + \bar{Y}^I)$$

Attractor equations: $\quad \partial_U\,\mathcal{E} = 0\;,\; \partial_\Upsilon\,\mathcal{E} = 0\;,\; \partial_{Y^I}\,\mathcal{E} = 0$

$\Longrightarrow \quad U = U(q,p)\;,\; \Upsilon = \Upsilon(q,p)\;,\; Y^I = Y^I(q,p)$

Non-BPS solutions: complicated equations

BPS solutions: $\quad \mathcal{P}^I = \mathcal{Q}_I = 0\;,\; U = v_1/v_2 = 1\;,\; \Upsilon = -64$

$\Longrightarrow$ Entropy function becomes $\quad \mathcal{E} = \Sigma$

Entropy Function for BPS Black Holes

BPS entropy function $\mathcal{E} = \Sigma$ can be used to define a duality invariant version of the OSV integral for BPS black holes

`C + de Wit + Käppeli + Mohaupt, hep-th/0601109`

$$\begin{aligned} d(q,p) &= \int d(Y+\bar{Y})^I \, d(F_I + \bar{F}_I) \, \mathrm{e}^{\pi \, \Sigma(Y,\bar{Y},q,p)} \\ &\propto \int dY \, d\bar{Y} \, \Delta(Y,\bar{Y}) \, \mathrm{e}^{\pi \, \Sigma(Y,\bar{Y},q,p)} \end{aligned}$$

where $\Delta = \det N_{IJ} = \det [\mathrm{Im}\, F_{IJ}]$. Measure factor.

Assuming N_{IJ} non-degenerate (i.e. large black holes),

$$d(q,p) \approx \mathrm{e}^{\pi \, \Sigma|_{\mathrm{extr}}} = \mathrm{e}^{\mathcal{S}_{\mathrm{macro}}(q,p)}$$

in saddle-point approximation.

Attractor Equations: Two-derivative Level

Without R^2-terms, entropy function is given by

$$\begin{aligned}\mathcal{E} &= U\,\left[\Sigma + \left(\mathcal{Q}_I - F_{IJ}\,\mathcal{P}^J\right) N^{IK}\left(\mathcal{Q}_K - \bar{F}_{KL}\,\mathcal{P}^L\right)\right] \\ &\quad + \frac{8i}{\sqrt{-\Upsilon}}\,(1-U)\,\left(\bar{Y}^I\,F_I - Y^I\,\bar{F}_I\right)\end{aligned}$$

Extremizing $\mathcal{E}$ yields with respect to Υ, U and Y^I yields

$$U = 1 \quad , \quad \sqrt{-\Upsilon} = 8\left(1 + \frac{\left(\mathcal{Q}_I - F_{IJ}\,\mathcal{P}^J\right) N^{IK}\left(\mathcal{Q}_K - \bar{F}_{KL}\,\mathcal{P}^L\right)}{\Sigma}\right)^{-1}$$

$$\begin{aligned}&-2\left(\mathcal{Q}_J - F_{JK}\,\mathcal{P}^K\right) - F_{IMJ}\,\mathcal{P}^M\,N^{IK}\left(\mathcal{Q}_K - \bar{F}_{KL}\,\mathcal{P}^L\right) \\ &+i\left(\mathcal{Q}_I - F_{IM}\,\mathcal{P}^M\right) N^{IR}\,F_{RSJ}\,N^{SK}\left(\mathcal{Q}_K - \bar{F}_{KL}\,\mathcal{P}^L\right) = 0\end{aligned}$$

for any extremal black hole solution. The entropy is given by

$$\mathcal{S}_{\rm macro} = \pi\,\mathcal{E}|_{\rm extr} = 8\pi\,\frac{\Sigma}{\sqrt{-\Upsilon}}|_{\rm extr}$$

Extremal Black Holes in $D = 5$

with Jan Perz and Johannes Oberreuter

BPS black holes in $D = 4$ are connected to five-dimensional BPS black holes in Taub-NUT spaces, with NUT charge $p^0 = 1$

Gaiotto + Strominger + Yin, hep-th/0503217

Connection based on cubic prepotentials $F(Y) = D_{ABC} Y^A Y^B Y^C / Y^0$ and the KK dictionary relating $z^A = Y^A / Y^0 \leftrightarrow \hat{X}^A$ in $D = 4$ and $D = 5$.

Expect this connection to also hold for non-BPS extremal black holes:

- define entropy function in $D = 5$ (real special geometry) in terms of entropy function in $D = 4$ (special geometry): $\mathcal{E}_5 = \mathcal{E}_4$

  For example: charges $(p^0 = 1, q_A \neq 0)$, schematically

$$\mathcal{E}_5 = q_A \, G^{AB}(\hat{X}) \, q_B + D_{ABC} \hat{X}^A \, \hat{X}^B \, \hat{X}^C$$

Extremal Black Holes in $D = 5$

- Chern-Simons terms in $D = 5$
- four-dimensional attractor equations for $(Y^0, Y^A) \Longrightarrow$ five-dimensional attractor equations for $\hat{X}^A$

Thanks!