Exact Supersymmetric Solutions of AdS/CFT

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Outline

1. $\mathcal{N} = 4$ SCFT and $AdS_5 \times S^5$
   - Kinematics of BPS Sector
   - Solutions of the SUGRA Equations

2. $\mathcal{N} = 1$ SCFT and $AdS_5 \times Y^{p,q}$
   - Vacuum Solutions
   - The simplest $Y^{p,q}$

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Kinematics of the Correspondence

The duality

$$\mathcal{N} = 4\; SU(N)\; SYM \text{ on } \mathbb{R} \times S^3 \cong \text{II B String Theory on } AdS_5 \times S^5$$

$$\frac{L_{AdS}^2}{4\pi\ell_s^2} = \left(\frac{\lambda}{4\pi}\right)^{1/2} \quad , \quad g_s = \frac{\lambda}{N} \qquad (\lambda \equiv g_{YM}^2 N)$$

Matching the spectra

We can consider solutions of the Supergravity equations instead of states in the String Theory provided

$$\mathcal{R}_4 L^4 \ll \lambda \ll N$$

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Gauge Theory BPS States

We consider lowest KK mode on S^3 of operators of the form

$$\mathrm{Tr}(X^{q/2})\mathrm{Tr}(Y^{q/2})\mathrm{Tr}(Z^r)$$

Supersymmetries

$q = 0, r \neq 0$	$q \neq 0, r = 0$	$q \neq 0, r \neq 0$
$1/2\ BPS$	$1/4\ BPS$	$1/8\ BPS$

BPS Bound

$$\Delta = q + r$$

Bosonic Symmetry

$$\mathbb{R}_{BPS} \times \left(SU(2)_L \times U(1)_R\right)_{R-\mathrm{charge}} \times SO(4)_{KK}$$

Supergravity BPS Solutions

Bosonic Symmetry

$$\mathbb{R}_{BPS} \times \left(SU(2)_L \times U(1)_R\right)_{R-\text{charge}} \times SO(4)_{KK}$$

- The Ansatz

$$ds^2 = \tilde{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \tilde{\rho}^2 \mathrm{d}\tilde{\Omega}_3^2 + \rho_1^2\left(\hat{\sigma}_1^2 + \hat{\sigma}_2^2\right) + \rho_3^2(\hat{\sigma}_3 - A_\mu \mathrm{d}x^\mu)^2$$

$$F_{(5)} = \mathcal{F}_5 + \star\mathcal{F}_5$$

$$\frac{1}{2}\mathcal{F}_5 = \left[\tilde{G}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + \tilde{V}_\mu \mathrm{d}x^\mu \wedge \left(\hat{\sigma}_3 - A_\mu \mathrm{d}x^\mu\right) + \tilde{g}\hat{\sigma}_1 \wedge \hat{\sigma}_2\right] \wedge \mathrm{d}\tilde{\Omega}_3$$

- The Supersymmetry Condition

$$\delta\chi_M = \nabla_M \psi + \frac{\mathrm{i}}{480} \not{F}\,\Gamma_M \psi = 0$$

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Supergravity BPS Solutions

- Bosonic Symmetries, Supersymmetries

The Solution

$$\mathrm{d}s^2 = -h^{-2}(\mathrm{d}t + V_i \mathrm{d}x^i)^2 + h^2 \frac{\rho_1^2}{\rho_3^2}(T^2 \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j + \mathrm{d}y^2) + \tilde{\rho}^2 \mathrm{d}\tilde{\Omega}_3^2 + \\ + \rho_1^2(\hat{\sigma}_1^2 + \hat{\sigma}_2^2) + \rho_3^2(\hat{\sigma}_3 - A_t \mathrm{d}t - A_i \mathrm{d}x^i)^2$$

$$y = \rho_1 \tilde{\rho} > 0$$

Everything can be expressed in terms of four functions of (x^1, x^2, y)

$$m, n, p, T$$

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Everything can be expressed in terms of four functions of (x^1, x^2, y)

$$m, n, p, T + \text{Bianchi Identities}$$

Differential equations

- Second order, Coupled, Non Linear Differential Equations
  $\Rightarrow$ *Boundary conditions*

$$(\partial_1^2+\partial_2^2)n+\frac{1}{y^3}\partial_y\left(y^3T^2\partial_y n\right)+\frac{1}{y}\partial_y\left[T^2\left(yDn+2y^2m(n-p)\right)\right]+ \\ +4y^2DT^2n=0$$

$$(\partial_1^2+\partial_2^2)m+\frac{1}{y^3}\partial_y\left(y^3T^2\partial_y m\right)+\partial_y\left(y^3T^2 2mD\right)=0$$

$$(\partial_1^2+\partial_2^2)p+\frac{1}{y^3}\partial_y\left(y^3T^2\partial_y p\right)+\partial_y\left[y^3T^2 4ny(n-p)\right]=0$$

$$\partial_y T=2yD\,T \qquad D\equiv m+n-1/y^2$$

HARD to solve!

Differential equations

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...but..

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Simplifying Condition

Lin, Lunin, Maldacena, hep-th/0409174

LLM limit

$$n - p = D = 0 \Rightarrow n = p = \frac{1}{y^2} - m \quad ; \quad T = 1$$

- The three equations collapse to a single one

$$(\partial_1^2 + \partial_2^2)n + \frac{1}{y^3}\partial_y \left(y^3 \partial_y n\right) = 0$$

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$$+ \rho_1^2(\hat{\sigma}_1^2 + \hat{\sigma}_2^2) + \rho_3^2(\hat{\sigma}_3 - A_t dt - A_i dx^i)^2$$

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- Enhanced Symmetry $\rightarrow \mathbb{R} \times SO(4)_R \times SO(4)_{KK}$
- Enhanced SUSY $\rightarrow \frac{1}{2} BPS \cong \mathrm{Tr}(Z^r)$

Leading Asymptotics

$AdS_5 \times S^5$

Boundary Conditions at infinity

$$m \sim \frac{1}{y^2} - \frac{1}{L^4 R^4}$$

$$n \sim \frac{1}{L^4 R^4} \quad , \quad p \sim \frac{1}{L^4 R^4}$$

$$T \sim 1$$

$$L^4 R^2 = x_1^2 + x_2^2 + y^2$$

$$\mathrm{d}s^2 \sim L^2 \left(-R^2 \mathrm{d}t^2 + \frac{\mathrm{d}R^2}{R^2} + R^2 \mathrm{d}\tilde{\Omega}_3^2 + \mathrm{d}\theta^2 + \cos^2\theta \mathrm{d}\tilde{\phi}^2 + \sin^2\theta \mathrm{d}\hat{\Omega}_3^2 \right)$$

Subleading Corrections

Constants of integration

Assume $\partial_{\tilde{\phi}}$ is a Killing vector [Note: this lifts the internal symmetry to $SU(2) \times U(1) \times U(1)$]

$$m \sim \frac{1}{y^2} - \frac{L^4}{R^4} + \frac{m_3(\theta)}{R^6}$$

$$n \sim \frac{L^4}{R^4} + \frac{n_3(\theta)}{R^6} \qquad p \sim \frac{L^4}{R^4} + \frac{p_3(\theta)}{R^6}$$

$$T \sim 1 + \frac{t_1(\theta)}{R^2} + \frac{t_2(\theta)}{R^4} .$$

- A priori there are seven constants of integration
- Coordinates redefinitions and Regularity reduce them to two

$$J \, , Q$$

Subleading Correction

KK Charges of the Solutions

Off diagonal components of the metric $\xrightarrow{KK}$ two $U(1)$ gauge fields
We can read off the $U(1)$ charges of the solutions from:

$$g_{t\phi}\mathrm{d}t\mathrm{d}\tilde{\phi} \sim g_{\phi\phi}\frac{J}{R^2}\mathrm{d}t\mathrm{d}\tilde{\phi} \qquad , \qquad g_{t\hat{3}}\mathrm{d}t\,\hat{\sigma}_3 \sim g_{\hat{3}\hat{3}}\frac{Q}{R^2}\mathrm{d}t\,\hat{\sigma}_3$$

BPS bound

$$M = \frac{\pi}{4G_5}(Q+J)$$

Gauge Theory

$$\mathrm{Tr}(X)^{q/2}\mathrm{Tr}(Y)^{q/2}\mathrm{Tr}(Z)^r \qquad \Delta = q + r$$

$$Q \cong q \quad J \cong r$$

Subleading Correction

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So far So good

- (at least) $\frac{1}{8}$ BPS Solutions of the SUGRA equations
- Beyond the probe approximation for D3-branes
- Perfect agreement for charges and mass

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More Supersymmetries

An interesting subset

- $SU(2) \times U(1) \times U(1)$ internal symmetry
- $\frac{1}{4} BPS$

$\Rightarrow$ They are of the form [Gauntlett, Martelli, Sparks, Waldram - hep-th/0403002]

$$ds^2 = \left(-\tilde{\rho}^2 dt^2 + \frac{L^2 d\tilde{\rho}^2}{\tilde{\rho}^2} + \tilde{\rho}^2 d\tilde{\Omega}_3^2 \right) \times L^2 ds^2 \left(Y^{p,q} \right)$$

- $Y^{p,q}$ are a class of five dimensional Sasaki- Einstein geometries
- Base of Calabi-Yau cones

Excitations of the $\mathcal{N} = 1$ vacuum

- $AdS_5 \times Y^{p,q}$ geometries are dual to vacuum states of an $\mathcal{N} = 1$ Quiver Gauge Theory [Benvenuti, Franco, Hanany, Martelli, Sparks - hep-th/0411264]
- We can study $\frac{1}{8}$ BPS solutions which are asymptotically $AdS_5 \times Y^{p,q}$
- They are dual to states of the Gauge Theory preserving one half of its supersymmetries

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Leading Asymptotics

$AdS_5 \times T^{1,1}$

The simplest $Y^{p,q}$ is $Y^{1,0} \equiv T^{1,1}$

Leading Asymptotics

$AdS_5 \times T^{1,1}$

Boundary Conditions at Infinity

$$m \sim \frac{3}{2}\frac{1}{y^2}$$

$$n \sim \frac{2}{3}\frac{L^4}{y^4} \qquad p \sim \frac{4}{9}\frac{L^4}{y^4}$$

$$T \sim \frac{y}{L^2}\sqrt{6}\sin(\theta)\left(\tan\frac{\theta}{2}\right)^{-1/6}$$

$$y^2 = \frac{2}{3}L^2\tilde{\rho}^2 \qquad x_1^2 + x_2^2 = L^4\left(\tan\frac{\theta}{2}\right)^{-1/3}$$

Leading Asymptotics

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$$ds^2 \sim \left(-\tilde{\rho}^2 dt^2 + \frac{L^2 d\tilde{\rho}^2}{\tilde{\rho}^2} + \tilde{\rho}^2 d\tilde{\Omega}_3^2 \right) +$$
$$+ L^2 \left[\frac{1}{6} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{1}{6} (d\hat{\theta}^2 + \sin \hat{\theta} d\hat{\phi}^2) + \frac{1}{9} (d\hat{\psi} + \cos \hat{\theta} d\hat{\phi} + \cos \theta d\phi)^2 \right]$$

First Subleading Corrections

J and *Q* charge

$$m \sim \frac{3}{2}\frac{1}{y^2} + \frac{m_2(\theta)}{y^4}$$

$$n \sim \frac{2}{3}\frac{L^4}{y^4} + \frac{n_3(\theta)}{y^6} \qquad p \sim \frac{4}{9}\frac{L^4}{y^4} + \frac{p_3(\theta)}{y^6}$$

$$T \sim \frac{y}{L^2}\sqrt{6}\sin(\theta)\left(\tan\frac{\theta}{2}\right)^{-1/6} + \frac{t_1(\theta)}{y}$$

Three integration constants out of seven

J, Q, B

J, Q still measure the charge w.r.t. KK gauge fields of the metric

First Subleading Corrections

Baryon number charge

- B measures the $F_{(5)}$ flux of D3-branes wrapped over a non trivial 3-cycle in $T^{1,1}$
- The KK reduction of $F_{(5)}$ over such a cycle gives rise to a third $U(1)_B$ gauge field B_μ
- B is proportional to the $U(1)_B$ charge of the solutions
- Such wrapped D3-branes are known to be dual to baryons in the gauge theory
- B is proportional to the baryon charge of our solutions

[Herzog, Klebanov, Ouyang - hep-th/0205100]

Note

Our solutions are supposed to be smooth and realize the geometrical transition of such D3 branes

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Summary

- Parametrization of exact Supergravity Solutions dual to
  - $\frac{1}{8}$ and $\frac{1}{4}$ BPS States in $\mathcal{N} = 4$ $SU(N)$ SYM
  - half supersymmetric states in $\mathcal{N} = 1$ Quiver gauge theories
- Beyond the probe approximation for D3 branes wrapped on non trivial cycles of $Y^{p,q}$
- Correct matching of charges

- Outlook
  - Identify boundary conditions at $y = 0$
  - Solve the differential equations
  - Map to the known (singular) solutions
  - Quantize the phase space of solutions
  - Study different $Y^{p,q}$ asymptotics
  - ...