Probe D-branes in Superconformal Field Theories

José D. Edelstein
Universidade de Santiago de Compostela

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Based on joint work with
Felipe Canoura, Alfonso V. Ramallo (USC), Leopoldo A. Pando Zayas, Diana Vaman (UMich)
Felipe Canoura, Alfonso V. Ramallo (USC)

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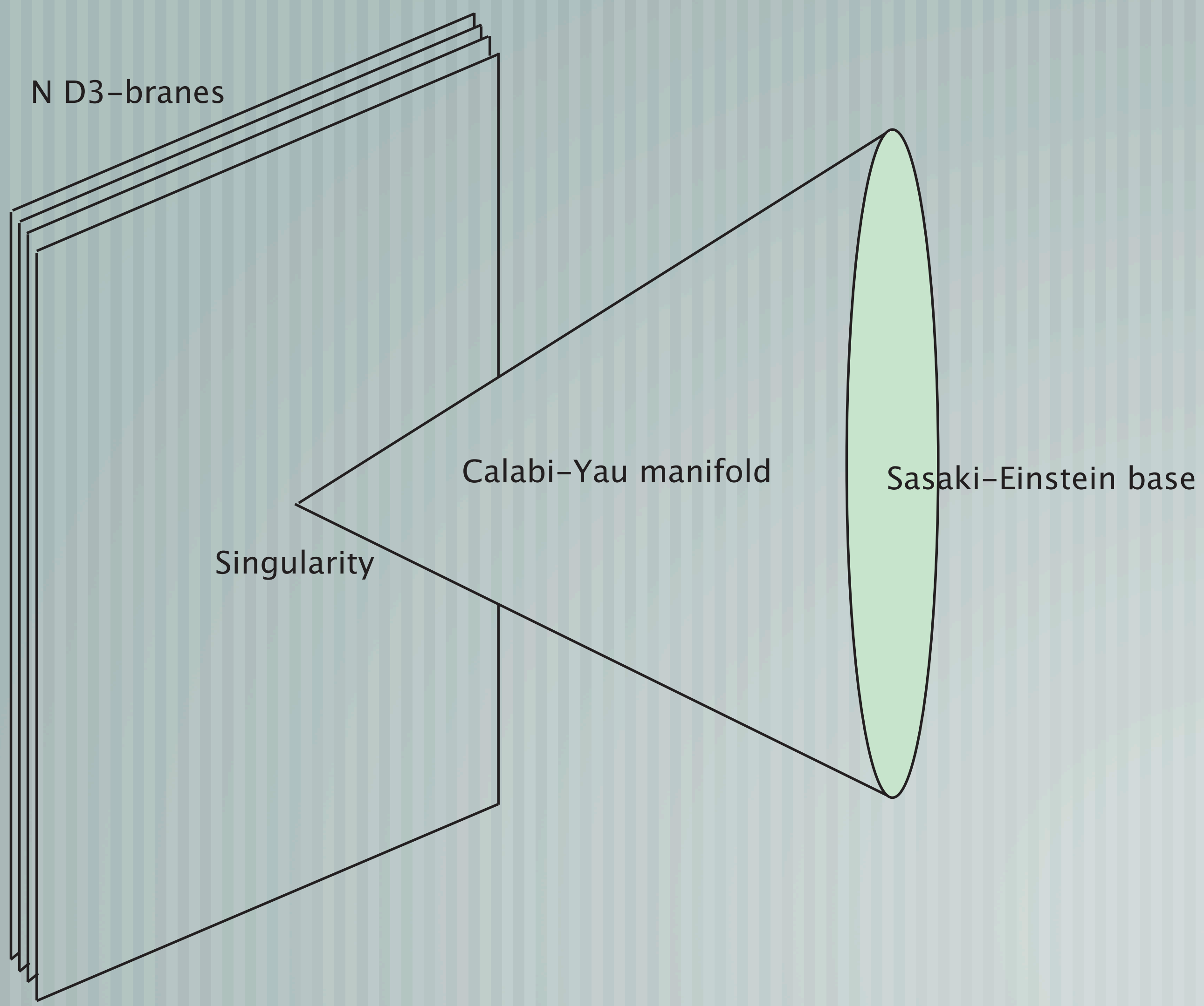
4d N=1 SCFT & AdS/CFT

Given a Sasaki–Einstein manifold X^5, a stack of N parallel D3–branes at the tip of the (Calabi–Yau) cone over X^5 leads to:

$\mathcal{N}=1$ superconformal quiver theory is dual to type IIB in $AdS_5 \times X^5$ — Gubser, 1998



$\Rightarrow$ $X^5 = S^5$ — Maldacena, 1997; Gubser-Klebanov-Polyakov, 1998; Witten, 1998

$\Rightarrow$ $X^5 = T^{1,1}$, $SU(2)\times SU(2)\times U(1)$ — Tanno, 1979; Romans, 1984; Candelas-De la Ossa, 1989; Klebanov-Witten, 1998

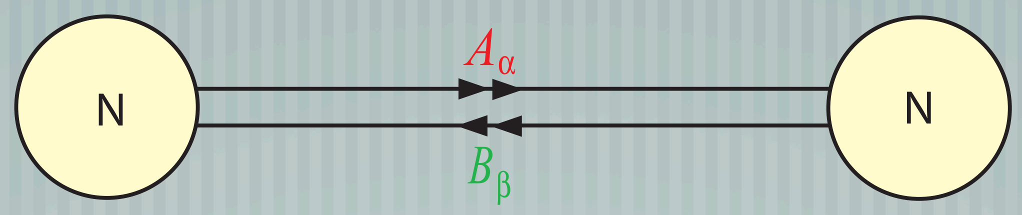
$\Rightarrow$ $X^5 = Y^{p,q}$, $SU(2)\times U(1)\times U(1)$ — Gauntlett-Martelli-Sparks-Waldram, 2004; Martelli-Sparks, 2004; Benvenuti-Franco-Hanany-Martelli-Sparks, 2004

$\Rightarrow$ $X^5 = L^{a,b,c}$, $U(1)\times U(1)\times U(1)$ — Cvetic-Lu-Pope, 2005; Martelli-Sparks, 2005; Benvenuti-Kruczenski, 2005; Franco-Hanany-Martelli-Sparks-Vegh-Wecht, 2005; Butti-Forcella-Zaffaroni, 2005

Klebanov-Witten model

When $X^5 \equiv T^{1,1}$ we have the Klebanov–Witten model dual to 4d $\mathcal{N}=1$ $SU(N)\times SU(N)$ SCFT with a flavor $SU(2)\times SU(2)$ coupled to two chiral superfields in the bifundamental representation

Klebanov-Witten, 1998



	$SU(N)\times SU(N)$	$SU(2)\times SU(2)$
A_α	$(\mathbf{N},\bar{\mathbf{N}})$	$(\mathbf{2},\mathbf{1})$
B_β	$(\bar{\mathbf{N}},\mathbf{N})$	$(\mathbf{1},\mathbf{2})$

with a superpotential

$$W=\epsilon^{ij}\,\epsilon^{kl}\ \mathrm{Tr}\ A_i\,B_k\,A_j\,B_l$$

$$ds_5^2\,(T^{1,1})=\frac{1}{6}\sum_{i=1}^{2}(d\theta_i^2+\sin^2\theta_i d\phi_i^2)+\left[\frac{1}{3}\,d\psi+\sigma\right]^2$$

The isometry group of $T^{1,1}$ is clearly $SU(2)\times SU(2)\times U(1)$

The topology is that of $S^2\times S^3$

$$U(1)\rightarrow T^{1,1}\rightarrow S^2\times S^2$$

$S^2\times S^2$ is a Kähler–Einstein four-manifold

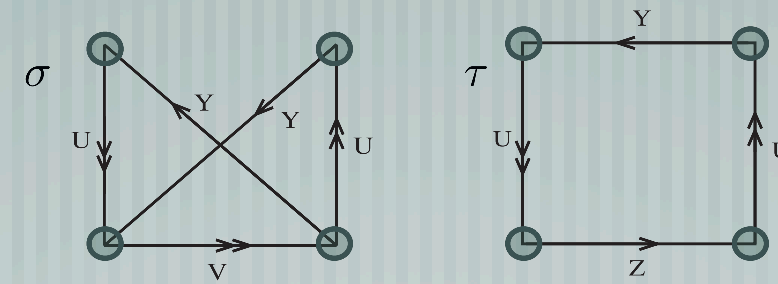
Quiver theories for Y ^{p,q}

Gauge group: $SU(N) \times \cdots \times SU(N)$ ($2p$ times)

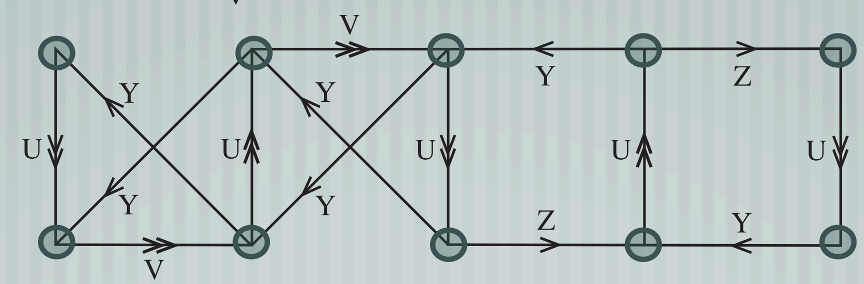
Benvenuti-Franco-Hanany-Martelli-Sparks, 2004

Four types of bifundamental chiral fields: $\begin{cases} U^\alpha \quad V^\beta & \text{(doublets)} \\ Y \quad Z & \text{(singlets)} \end{cases}$ (of the global $SU(2)$)

The quiver theory for $Y^{p,q}$ can be constructed from two basic cells denoted by σ and τ



$$Y^{4,2} \equiv \sigma\tilde{\sigma}\tau\tilde{\tau}$$



$\tilde{\sigma}$ and $\tilde{\tau}$ are the mirror images with respect to a horizontal axis

$Y^{p,q}$ quivers are built with q σ and $p-q$ τ unit cells. The terms in the superpotential come from closed loops

$$W = \sum_{i=1}^{q} \epsilon_{\alpha\beta}(U_i^\alpha V_i^\beta Y_{2i-1} + V_i^\alpha U_{i+1}^\beta Y_{2i}) + \sum_{j=q+1}^{p} \epsilon_{\alpha\beta} Z_j U_{j+1}^\alpha Y_{2j-1} U_j^\beta$$

i, j refer to the gauge group where the arrow originates

Quantum numbers

The global $U(1)$ symmetries in the isometry group are identified as $U(1)_R$ and a flavor symmetry $U(1)_F$. There is also a baryonic $U(1)_B$ that becomes a gauge symmetry in the gravity dual.

Field	number	$R-$ charge	$U(1)_B$	$U(1)_F$
Y	$p+q$	$\frac{-4p^2+3q^2+2pq+(2p-q)\sqrt{4p^2-3q^2}}{3q^2}$	$p-q$	-1
Z	$p-q$	$\frac{-4p^2+3q^2-2pq+(2p+q)\sqrt{4p^2-3q^2}}{3q^2}$	$p+q$	$+1$
U^α	p	$\frac{2p(2p-\sqrt{4p^2-3q^2})}{3q^2}$	$-p$	0
V^β	q	$\frac{3q-2p+\sqrt{4p^2-3q^2}}{3q}$	q	$+1$

The above assignment satisfies a number of conditions: the linear anomalies vanish $\mathrm{Tr}\, U(1)_B = \mathrm{Tr}\, U(1)_F = 0$, as well as the cubic t' Hooft anomalies $\mathrm{Tr} U(1)_B^3$ and $\mathrm{Tr} U(1)_F^3$.

D-branes in the gravity side

The gravity side **must contain D–branes** — Witten, 1998

There are several features of the gauge theory that demand the introduction of (wrapped) D–branes in the gravity side:

- The baryon vertex – a baryon built out of external quarks
  D5-brane wrapping the whole five-dimensional compact manifold
- Domain walls – fractional branes or defect CFT
  D5–branes wrapping 2-cycles of the internal geometry
- Dibaryon operators – built from chiral fields in quiver theories
  D3–branes wrapping supersymmetric 3-cycles
- Matter hypermultiplets – quarks in the fundamental representation — Karch-Katz, 2002
  Spacetime filling wrapped D7–brane

Many of these aspects can be studied at the probe approximation level

For example, the introduction of matter in the quenched approximation $N_f \ll N_c$

D-brane probes

Consider Dp-brane probes in $AdS_5 \times X^5$. The embedding is characterized by the set of functions $X^M(\xi^\mu)$.

The supersymmetric embeddings of the brane probes are obtained by imposing the kappa-symmetry condition:

Becker-Becker-Strominger, 1995
Bergshoeff-Kallosh-Ortín-Papadopoulos, 1997
Bergshoeff-Townsend, 1999

$$\Gamma_\kappa \, \epsilon \, = \, \epsilon$$

where ϵ is a Killing spinor of the background and Γ_κ is a matrix that depends on the embedding.

Cederwall-von Gussich-Nilsson-Sundell-Westerberg, 1996
Bergshoeff-Townsend, 1996
Aganagic-Popescu-Schwarz, 1996

$$\Gamma_\kappa \, = \, \frac{1}{(p+1)!\sqrt{-g}} \, \epsilon^{\mu_1 \cdots \mu_{p+1}} \, (\tau_3)^{\frac{p-3}{2}} \, i\tau_2 \, \otimes \, \gamma_{\mu_1 \cdots \mu_{p+1}}$$

(in the absence of worldvolume gauge fields). The Killing spinors in Sasaki-Einstein X^5 manifolds read $(\Gamma_* \equiv i\Gamma_{x^0 x^1 x^2 x^3})$

$$\epsilon \, = \, e^{-\frac{i}{2}\tilde{\psi}} \, r^{-\frac{\Gamma_*}{2}} \, \Big(1 \, + \, \frac{\Gamma_r}{2L^2} \, x^\alpha \, \Gamma_{x^\alpha} \, (1 \, + \, \Gamma_*) \Big) \, \eta \qquad \begin{cases} \Gamma_{12} \, \eta \, = \, -i\eta \\ \Gamma_{34} \, \eta \, = \, i\eta \end{cases}$$

$\Gamma_\kappa \, \epsilon = \epsilon$ imposes a new projection to the Killing spinor and give rise to a set of first-order BPS differential equations. They determine the supersymmetric embeddings of the brane probes.

It is a local condition that must be satisfied *at any point of the probe worldvolume.*

Aspects of $Y^{p,q}$

The metric of the Sasaki-Einstein space $Y^{p,q}$ can be written as

$$ds^2_{Y^{p,q}} = \frac{1-cy}{6}\underbrace{(d\theta^2 + \sin^2\theta\, d\phi^2)}_{S^2 \equiv SU(2)} + \frac{1}{6\,H^2(y)}dy^2 + \frac{\mathcal{Q}(y)}{9(a-y^2)}(\overset{\downarrow U(1)}{d\psi} + \cos\theta d\phi)^2 + \frac{2(a-y^2)}{1-cy}\left[\overset{\downarrow U(1)}{d\alpha} + \frac{ac-2y+y^2c}{6(a-y^2)}(d\psi + \cos\theta d\phi)\right]^2$$

$$H(y) = \sqrt{\frac{\mathcal{Q}(y)}{3(1-cy)}} \qquad \mathcal{Q}(y) = a - 3y^2 + 2cy^3 = 2c\prod_{i=1}^{3}(y-y_i)$$

Isometry group: $SU(2)\times U(1)\times U(1)$
(global symmetry in the field theory side)

- If $c=0$, we recover $T^{1,1}$
- If $c\neq 0$, we can set $c=1$ and the metric is regular iff, in terms of two coprime integers $p>q$, $\quad a = \frac{1}{2} - \frac{p^2-3q^2}{4p^3}\sqrt{4p^2-3q^2}$

For this value of a, the coordinates range results $\quad y_1 \leq y \leq y_2 \quad 0\leq\theta\leq\pi \quad 0\leq\phi\leq 2\pi \quad 0\leq\alpha\leq 2\pi\ell \quad 0\leq\psi\leq 2\pi$
where y_1, y_2, ℓ are specific irrational functions of p and q

The quantization condition of the flux of $F^{(5)}$ $\quad L^4 = \frac{4\pi^4}{\mathrm{Vol}(Y^{p,q})}g_s N(\alpha')^2 \qquad \mathrm{Vol}(Y^{p,q}) = \frac{q^2}{3p^2}\frac{2p+\sqrt{4p^2-3q^2}}{3q^2-2p^2+p\sqrt{4p^2-3q^2}}\pi^3$

Aspects of CY^{p,q}

In order to clarify some aspects of $Y^{p,q}$, $\beta = -(6\alpha + c\psi)$, and the canonical form (notice that β is not a periodic coordinate)

$$ds^2_{Y^{p,q}} = \frac{1-cy}{6}\,(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{1}{6\,H^2(y)}\,dy^2 + \frac{H^2(y)}{6}\,(d\beta - c\cos\theta d\phi)^2 + \frac{1}{9}\left[d\psi + \cos\theta d\phi + y(d\beta - c\cos\theta d\phi)\right]^2$$

It neatly displays local features of these spaces: $$ds^2_{Y^{p,q}} = ds^2_4 + \left[\frac{1}{3}d\psi + \sigma\right]^2$$

where ds^2_4 is locally a Kähler–Einstein metric with Kähler form $J_4 = \frac{1}{2}d\sigma$

It is possible to define a set of local complex coordinates on $CY^{p,q}$

$$z_1 = \tan\frac{\theta}{2}\, e^{-i\phi} \qquad z_2 = (\sin\theta)^c \exp\left(-\int \frac{1}{H(y)^2}dy\right) e^{-i\beta} \qquad z_3 = r^3 \sin\theta \exp\left(-\int \frac{y}{H(y)^2}dy\right) e^{i\psi}$$

They are meromorphic functions on $CY^{p,q}$: z_1 (z_2) is singular at $\theta = \pi$ ($y = y_1$) and z_2 is not globally well-defined

The holomorphic three-form of $CY^{p,q}$ simply reads

$$\Omega = -\frac{1}{18\sqrt{3}}\frac{dz_1 \wedge dz_2 \wedge dz_3}{z_1 z_2}$$

D3-branes on singlet 3-cycles...

A singlet object spans θ and ϕ coordinates.

Take $\xi^{\mu} = (\tau, \theta, \phi, \beta)$ and a generic embedding $y(\theta, \phi, \beta)$ and $\psi(\theta, \phi, \beta)$. The kappa symmetry matrix:

$$\Gamma_{\kappa}\epsilon = \frac{\cosh\rho}{\sqrt{-g}}\,\Gamma_{\tau}\left[\,a_5\,\Gamma_5 + a_1\Gamma_1 + a_3\Gamma_3 + a_{135}\,\Gamma_{135}\,\right]\epsilon \equiv \epsilon$$

where the coefficients on the r.h.s. involve the background, y and ψ, and their first order derivatives. E.g.

$$a_{135} = \frac{\sqrt{1-cy}}{18}\left[\frac{\sin\theta}{H}\left[\psi_{\theta}y_{\beta} - (y+\psi_{\beta})\,y_{\theta}\,\right] + H\left[\psi_{\phi} + (1+c\psi_{\beta})\cos\theta\,\right] + \frac{i}{H}\left[(\psi_{\phi} + (1-cy)\,\cos\theta\,)y_{\beta} - (y+\psi_{\beta})\,y_{\phi}\right] - iH\sin\theta\psi_{\theta}\right]$$

The matrices Γ_1, Γ_3 and Γ_{135} do not commute with SUSY projections. Thus, we must impose $a_1 = a_3 = a_{135} = 0$:

$$H(y) = 0 \qquad \text{i.e.} \quad y = y_1 \quad \mathit{or} \quad y = y_2$$

Compatibility with the AdS_5 structure of the spinor implies that the D3-brane must be placed at the center of AdS_5, $\rho = 0$.

Therefore, if we place the D3-brane at the center of the AdS_5 space and wrap it on the three-cycles at $y = y_1$ or $y = y_2$, we obtain a $\frac{1}{8}$ supersymmetric configuration.

These configurations should correspond to (di)baryons in the gauge theory side.

Witten, 1998
Gubser-Klebanov, 1998
Gukov-Rangamani-Witten, 1998
Berenstein-Herzog-Klebanov, 2002
Beasley, 2002

...and their field theory duals

The dictionary of AdS/CFT tells us that $\Delta = LM$, M is the mass of the wrapped D3-brane, $M = T_3\, V_3$,

$$\frac{1}{T_3} = 8\pi^3(\alpha')^2 g_s \qquad V_3 = \int_{\mathcal{C}} \sqrt{g_{\mathcal{C}}}\; d^3\xi$$

$g_{\mathcal{C}}$ is the determinant of the spatial part of the induced metric on the 3-cycle $\mathcal{C}$

For the singlet cycles S_i at $y = y_i$ $(i = 1, 2)$ $\quad(\lambda_1 = +1,\ \lambda_2 = -1)$:

$$\Delta_i^{\mathrm{S}} = \frac{N}{2q^2}\Big[-4p^2 + 3q^2 + 2\lambda_i\, pq + (2p - \lambda_i\, q)\sqrt{4p^2 - 3q^2}\,\Big]$$

Being BPS saturated objects, R-charges are just $R_i = \frac{2}{3}\,\Delta_i^{\mathrm{S}}$, precisely as the operators $\det(Y)$ and $\det(Z)$

The baryon number is identified with the third homology class of $\mathcal{C}$ which, in units of N, is given by

$$\mathcal{B}(\mathcal{C}) = \pm i \int_{\mathcal{C}} P\Big[\,K\Big(\frac{dr}{r} + \frac{i}{L}\,e^5\Big) \wedge \omega\Big]_{\mathcal{C}}$$

ω is the self-dual 2-form and K is a constant. Then:

$$\mathcal{B}(\mathrm{S}_1) = -i\int_{\mathrm{S}_1} P\big[\,\Omega_{2,1}\big]_{\mathrm{S}_1} = p - q \qquad\qquad \mathcal{B}(\mathrm{S}_2) = i\int_{\mathrm{S}_2} P\big[\,\Omega_{2,1}\big]_{\mathrm{S}_2} = p + q$$

in perfect agreement with the baryon numbers of Y and Z!

D3-branes on doublet 3-cycles

We explore supersymmetric embeddings of the form $\xi^\mu = (\tau, y, \beta, \psi)$ with $\theta(y, \beta, \psi)$ and $\phi(y, \beta, \psi)$.

The simplest solution to kappa symmetric constraint is θ = constant and ϕ = constant

We can compute the conformal dimension:

$$\Delta^D = N \frac{p}{q^2} \left(2p - \sqrt{4p^2 - 3q^2} \right)$$

and the baryon number,

$$\mathcal{B}(\mathrm{D}) = -i \int_{\mathrm{D}} P\Big[K \left(\frac{dr}{r} + \frac{i}{L} e^5 \right) \wedge \omega \Big]_{\mathrm{D}} = -p$$

which lead to the identification of a dibaryon constructed with U^α

Herzog-Ejaz-Klebanov, 2004

It is possible to show that the BPS system can be written as Cauchy–Riemann equations for the above defined z_1 and z_2

Thus, they can be integrated in general with the result $z_2 = g(z_1)$

in agreement with the naive expectation that –locally!– they should determine a holomorphic embedding.

These are nontrivial kappa symmetric embeddings of a probe D3-brane on $AdS_5 \times Y^{p,q}$ but, for $c \neq 0$, they do not correspond to a wrapped D3-brane!

For $T^{1,1}$, instead, some of these embeddings correspond to interesting operators in the gauge theory

Areán-Crooks-Ramallo, 2004

Unfortunately, the relation between homogeneous coordinates and the chiral fields of the quiver theory is not as clear for $CY^{p,q}$

BPS fluctuations of dibaryons

Consider a dibaryon which is a singlet under $SU(2)$, say, $\det Y$. To construct excited dibaryons we should replace one of the Y factors, for example, by $YU^\alpha V^\beta Y$. We get a new operator of the form

$$\epsilon_1 \epsilon^2 (YU^\alpha V^\beta Y) Y \cdots Y$$

where ϵ_1 and ϵ^2 are completely anti-symmetric tensors for the $SU(N)$ factors. Using the identity $\epsilon^{a_1 \cdots a_N} \epsilon_{b_1 \cdots b_N} = \sum_\sigma (-1)^\sigma \delta^{a_1}_{\sigma(b_1)} \cdots \delta^{a_N}_{\sigma(b_N)}$ the new operator factorize into the original dibaryon and a single-trace operator

$$\mathrm{Tr}(U^\alpha V^\beta Y) \det Y$$

Excitations of a singlet dibaryon can be represented as graviton fluctuations in the presence of the dibaryon.

Instead, for the case of dibaryon with $SU(2)$ quantum number the situation is different. Consider, for simplicity, the state with maximum J_3 of the $SU(2)$

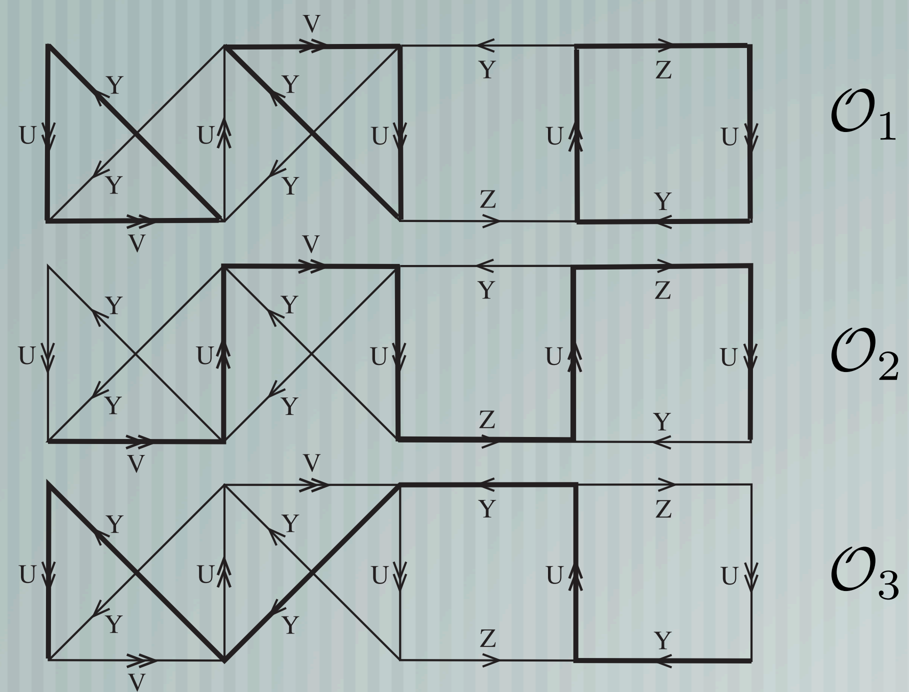
$$\epsilon_1 \epsilon^2 (U^1 \cdots U^1) = \det U^1$$

If the $SU(2)$ index of the U field is changed in the excitation, i.e. $U^1 \to U^2 \mathcal{O}$, where $\mathcal{O}$ is mesonic, the resulting operator cannot be decomposed as before.

Instead it has to be interpreted as a single particle state in AdS identified with a BPS excitation of the wrapped D3-brane corresponding to the dibaryon.

Mesonic chiral operators

The simplest ones, $\mathcal{O}_1$, are operators with R-charge 2, given by short loops of length 3 or 4 in the quiver (e.g. UVY, VUY or $YUZU$). It is a spin 1 chiral operator with scaling dimension $\Delta = 3$. Its $U(1)_F$ charge vanishes. There are also two classes of long loops in the quiver: $\mathcal{O}_2$ (e.g. $VUVUZUZU$) and $\mathcal{O}_3$ (e.g. $YUYYYU$) with spin, respectively, $\frac{p+q}{2}$ and $\frac{p-q}{2}$. They have a nonvanishing value of Q_F. The baryonic charge vanishes for any of these loops. E.g. $Y^{4,2}$



Operator	Q_R	Q_F	Spin
$\mathcal{O}_1$	2	0	1
$\mathcal{O}_2$	$p+q-\frac{1}{3\ell}$	p	$\frac{p+q}{2}$
$\mathcal{O}_3$	$p-q+\frac{1}{3\ell}$	$-p$	$\frac{p-q}{2}$

These are building blocks of all other scalar BPS operators, $\mathcal{O} = \prod_{i=1}^{3} \mathcal{O}_i^{n_i}$.

The spectrum of fluctuations of a dibaryon –that we obtain via an analysis of open string fluctuations on wrapped D3-branes– coincides with the mesonic chiral operators!

Further results with D3-branes

- We fully extended the present analysis to comprise the case of $X^5 = L^{a,b,c}$.

- We have showed that the cone on the 3-cycles wrapped by the D3–branes are calibrated submanifolds, i.e. divisors of $CY^{p,q}$ and $CL^{a,b,c}$.

$$P\Big[\frac{1}{2} J \wedge J\Big]_{\mathcal{D}} = \mathrm{Vol}(\mathcal{D})$$

- It would be interesting to understand the new family of supersymmetric embeddings of D3-branes in terms of operators in the field theory. It is worth stressing that global homogeneous coordinates exist in any toric variety but the relation to the field theory operators is much harder in $CY^{p,q}$ or $CL^{a,b,c}$.

- Another case of interest that we have considered is a probe D3-brane extended along one spatial direction of the gauge theory and wrapping a 2-cycle. By means of kappa symmetry we found that this embedding is not supersymmetric.

- Nevertheless, the Euler-Lagrange equations can be solved and the solutions is stable and represents a “fat string” from the gauge theory point of view.

D5-brane probes

- The embedding that we paid the most attention to is a D5-brane wrapping a two-dimensional submanifold in $Y^{p,q}$ and having codimension one in AdS_5. In the field theory this is the kind of brane that represents a domain wall across which the rank of the gauge groups jumps.
- Alternatively, if we allow the D5-brane to extend infinitely in the holographic direction, we would get a configuration dual to a defect CFT that preserves four supersymmetries.
- For this configuration we also considered turning on a worldvolume flux and found that it can be done in a supersymmetric way. The flux in the worldvolume of the brane provides a bending of the profile of the wall.
- We also considered D5-branes wrapping the whole $Y^{p,q}$, which corresponds to **the baryon vertex**. We verified that, as in the case of $T^{1,1}$, it is not a supersymmetric configuration.

  If a D5–brane wraps the whole $Y^{p,q}$ space, the flux of the RR $F^{(5)}$ acts as a source for the electric worldvolume gauge field which, in turn, gives rise to a bundle of F1s emanating from the D5-brane
- We fully extended the present analysis to comprise the case of $X^5 = L^{a,b,c}$.
- We have also showed that the cone on the 2-cycles wrapped by the D5–branes are calibrated (i.e. special Lagrangian) submanifolds of $CY^{p,q}$ and $CL^{a,b,c}$.

$$P\,[\,\Omega\,]_{\mathcal{L}} \,=\, e^{i\lambda}\,\mathrm{Vol}(\mathcal{L})$$

D7-brane probes

- With the aim of introducing mesons, we considered spacetime filling D7-branes: $\xi = (t, x^1, x^2, x^3, y, \beta, \theta, \phi)$

  and considered an embedding of the form $\psi = \psi(\beta, \phi)$ $\quad r = r(y, \theta)$

  In order to implement $\Gamma_\kappa \epsilon = \epsilon$, we must require $\Gamma_* \epsilon = -\epsilon$ and

$$r_y = \frac{r}{3H^2}\left(y + \psi_\beta\right) \qquad r_\theta = -\frac{r}{3\sin\theta}\left[(1 + c\psi_\beta)\cos\theta + \psi_\phi\right]$$

- These configurations preserve the four ordinary supersymmetries of the background

  For consistency with the assumed dependence of the functions of the ansatz, ψ_ϕ and ψ_β must be constants

$$\psi_\phi = n_1 \qquad \psi_\beta = n_2 \quad \Longrightarrow \quad \psi = n_1\,\phi + n_2\,\beta + \text{constant}$$

  It is now possible to obtain the function $r(\theta, y)$ $\quad r^3(y,\theta) = C\,\dfrac{\left[f_1(y)\right]^{n_2} f_2(y)}{\left[\sin\frac{\theta}{2}\right]^{1+n_1+cn_2}\left[\cos\frac{\theta}{2}\right]^{1-n_1+cn_2}}$

- This can be written as a holomorphic embedding $z_1^{m_1}\, z_2^{m_2}\, z_3^{m_3} = k$ $\quad \left(n_1 = \frac{m_1}{m_3} \quad n_2 = \frac{m_2}{m_3} \quad m_3 \neq 0\right)$

- We also analyzed a D7-brane that wraps $Y^{p,q}$ and is codimension two in AdS_5, a configuration that looks, from the field theory point of view, as a string that preserves two supercharges.

- We fully extended the present analysis to comprise the case of $X^5 = L^{a,b,c}$.

Conclusions and Final Comments

- We have found a large spectrum of supersymmetric wrapped D-branes (and also non-supersymmetric but stable branes) in $AdS_5 \times Y^{p,q}$ and $AdS_5 \times L^{a,b,c}$.

  These families exhaust all possible toric Calabi-Yau cones on a base with topology $S^2 \times S^3$.

- It would be interesting to work out the meson spectrum of these theories from the excitations of the spacetime filling D7-branes. We hope that understanding the conformal case might shed some light towards a better understanding of these issues in the context of AdS/CFT.

- It has been recently shown that the probe brane analysis can be smartly used as a key starting point to introduce flavor in the supergravity dual of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory (Maldacena-Núñez solution) beyond the probe approximation, *i.e.* for $N_f \sim N_c$.

  Casero-Paredes-Núñez, 2006

- The identification of supersymmetric four-cycles that a D7-brane can wrap in terms of local complex coordinates of the Calabi-Yau is relevant in cosmological models where inflation is produced by the motion of a D3-brane in a warped throat. The potential ruling this motion, in presence of a wrapped D7-brane, has being recently worked out.

  Baumann-Dymarsky-Klebanov-Maldacena-McAllister-Murugan, 2006

  The superpotential correction is actually given by the embedding equation that specifies the four-cycle, in agreement with a proposal made a decade ago.

  Ganor, 1996