# Supersymmetric D-branes on flux backgrounds

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Based on:

- L. M. & P. Smyth, hep-th/0507099
- L. M., hep-th/0602129
- P. Koerber & L. M., hep-th/0610044

# Outline

# Introduction

D-branes on backgrounds with reduced supersymmetry play a central role in many string theory models.

In $CY_3$ compactifications to four dimensions ($\mathcal{N} = 2$), D-brane physics is (relatively) well understood $\rightarrow$ key role of the underling integrable complex and Kähler structures.

In $\mathcal{N} = 1$ flux compactifications the CY's geometrical properties are generically lost and with them the related D-brane properties.

Question addressed in this talk:
Is it possible to describe (some of) the properties of D-branes on Type II $\mathcal{N} = 1$ backgrounds, keeping the analysis on very general grounds?

# A brief introduction on calibrations

A calibration in a certain supersymmetric background contains informations about the supersymmetric branes the background admits:

- Supersymmetric branes on purely geometric backgrounds (like CY spaces) are naturally *volume minimizing* [Becker, Becker & Strominger, '95] and then calibrated in the standard sense of [Harvey & Lawson, '82]. A calibration is a $p$-form $\omega_{(p)}$ such that

$$d\omega_{(p)} = 0 \quad \text{and} \quad P_\Sigma[\omega_{(p)}] \leq \sqrt{P_\Sigma[g]}d^p\sigma \text{ for any } p\text{-submanifold } \Sigma$$

- For branes with minimal action $\int\sqrt{-g} + \int A$ on backgrounds with nontrivial flux $F = dA$, the background calibration is naturally *energy minimizing* [Gutowski, Papadopoulos & Townsend, '99]. This notion has been used and extended e.g. by [Gauntlett, Kim, Martelli, Waldram, Pakis, Sparks, Cascales, Uranga,...]

D-branes contains a world-volume field-strength $\mathcal{F}$ (such that $d\mathcal{F} = P[H]$). The notion of calibration requires a further generalization for more general background and D-brane flux-configurations!

# Generalized calibrations for generalized cycles

[See also P. Koerber, hep-th/0506154]

- For a D-brane with energy density $\mathcal{E}$, we define a *generalized calibration* on our internal manifold as a polyform $\omega = \sum_k \omega_{(k)}$ such that
  - Algebraic condition:
    $P_\Sigma[\omega] \wedge e^{\mathcal{F}}|_{\text{top}} \leq \mathcal{E}(\Sigma, \mathcal{F})$ , for any generalized cycle $(\Sigma, \mathcal{F})$
  - Differential condition: $d_H \omega \equiv (d + H\wedge)\omega = 0$ .
- A D-brane wraps a *generalized calibrated cycle* $(\Sigma, \mathcal{F})$ iff

$$P_\Sigma[\omega] \wedge e^{\mathcal{F}}|_{\text{top}} = \mathcal{E}(\Sigma, \mathcal{F}) \,.$$

- A D-brane wrapping a generalized calibrated cycle $(\Sigma, \mathcal{F})$ is then energy minimizing under continuous deformations, i.e. for any $(\Sigma', \mathcal{F}')$ continuously connected to $(\Sigma, \mathcal{F})$

$$E(\Sigma, \mathcal{F}) \leq E(\Sigma', \mathcal{F}')$$

# Background ansatz

General Type II vacua preserving 4d Poincaré invariance and 4d $\mathcal{N} = 1$ supersymmetry:

$$\begin{aligned} &\text{metric:} && ds^2 = e^{2A(y)} dx^\mu dx_\mu + \dots \\ &\text{Killing spinors:} && \varepsilon_1(y) = \zeta_+ \otimes \eta_+^{(1)}(y) + \text{ c. c.} \\ & && \varepsilon_2(y) = \zeta_+ \otimes \eta_\mp^{(2)}(y) + \text{ c. c.} \end{aligned} \tag{1}$$

Introduce the polyforms $\hat{\Psi}_1 = \hat{\Psi}^\mp$ and $\hat{\Psi}_2 = \hat{\Psi}^\pm$ in IIA/IIB defined by Clifford associated bispinors

$$\eta_+^{(1)} \otimes \eta_\pm^{(2)\dagger} \sim \sum_{k=even/odd} \frac{1}{k!} \hat{\Psi}^\pm_{m_1 \dots m_k} \hat{\gamma}^{m_1 \dots m_k} \quad \leftrightarrow \quad \hat{\Psi}^\pm = \sum_{n=even,odd} \hat{\Psi}^\pm_{(n)}$$

The supersymmetry condition can be completely written in terms of equations for $\hat{\Psi}_1$ and $\hat{\Psi}_2$ [Graña, Minasian, Petrini & Tomasiello, hep-th/0505212].

# $\mathcal{N} = 1$ background supersymmetry and calibrations

We restrict to *D-calibrated* backgrounds, i.e. $||\eta^{(1)}|| = ||\eta^{(2)}||$ $\rightarrow$ most general $\mathcal{N} = 1$ backgrounds admitting static supersymmetric D-branes

Explicit form of the calibrations:

$$\begin{aligned} \omega^{(4d)} &= e^{4A}\left(e^{-\Phi}\mathrm{Re}\hat{\Psi}_1 - \tilde{C}\right) && \text{space-time filling branes} \\ \omega^{(\mathrm{string})} &= e^{2A-\Phi}\mathrm{Im}\hat{\Psi}_1 && \text{strings} \\ \omega^{(\mathrm{DW})} &= e^{3A-\Phi}\mathrm{Re}(e^{i\theta}\hat{\Psi}_2) && \text{domain walls} \end{aligned}$$

They satisfy the algebraic condition for generalized calibrations.

Differential condition $d_H\omega = 0 \Leftrightarrow$ background Killing spinor conditions!

$\kappa$-symmetry $\Rightarrow$ *Supersymmetric D-branes wrap calibrated generalized cycles*

For example, in the Calabi-Yau subcase the generalized calibrations are $\omega^{(\mathrm{even})} = \mathrm{Re}\left(e^{i\theta}e^{-iJ}\right)$, $\omega^{(\mathrm{odd})} = \mathrm{Re}\left(e^{i\theta}\Omega\right)$, and the calibration condition reproduces the supersymmetry conditions found by [Mariño, Minasian, Moore & Strominger, '99]

# Relation with Hitchin's and Gualtieri's generalized complex geometry

[Graña, Minasian, Petrini & Tomasiello, hep-th/0505212]

From domain wall calibrations we learn that

$$d_H\left(e^{3A-\Phi}\hat{\Psi}_2\right) = 0$$

$$\Downarrow$$

Since $\hat{\Psi}_2$ is a pure spinor, the associated *generalized complex structure* $\mathcal{J}_2$ is integrable $\Rightarrow$ the internal manifold $M$ is a Hitchin's *generalized Calabi-Yau*

$\hat{\Psi}_1$ is also pure but the RR-fields provide an obstruction to the integrability of the associated generalized almost complex structure $\mathcal{J}_1$.

# F and D-terms from the effective action

For a space-time filling D-brane wrapping a generalized $n$-cycle $(\Sigma, \mathcal{F})$ define

$$\begin{aligned} \mathcal{W}_m d\sigma^1 \wedge \ldots \wedge d\sigma^n &= P_\Sigma[e^{3A-\Phi}(\iota_m + g_{mk} dy^k \wedge)\hat{\Psi}_2] \wedge e^{\mathcal{F}}|_{\text{top}} \,, \\ \mathcal{D} d\sigma^1 \wedge \ldots \wedge d\sigma^n &= P_\Sigma[e^{2A-\Phi} \text{Im}\hat{\Psi}_1] \wedge e^{\mathcal{F}}|_{\text{top}} \,. \end{aligned}$$

The D-brane (with the appropriate orientation) is supersymmetric (i.e. calibrated) iff

$$\begin{aligned} \mathcal{W}_m = 0 \quad &, \qquad \text{F} - \text{flatness} \,, \\ \mathcal{D} = 0 \quad &, \qquad \text{D} - \text{flatness} \,. \end{aligned}$$

The identification $\mathcal{W}_m$ and $\mathcal{D}$ as F and D-terms comes from the expansion of DBI+CS action and susy transformations of the fermions around a susy configuration.

Furthermore, note that

F-flatness $\Leftrightarrow$ $(\Sigma, \mathcal{F})$ is a generalized complex submanifold

Simplest examples: Lagrangian and holomorphic cycles with $\mathcal{F}_{0,2} = 0$ are generalized complex submanifolds in symplectic and complex spaces respectively.

# The superpotential

The superpotential is given by

$$\mathcal{W}(\Sigma,\mathcal{F}) = \int_{\mathcal{B}} P[e^{3A-\Phi}\hat{\Psi}_2]\wedge e^{\tilde{\mathcal{F}}} + \text{ constant },$$

where $(\mathcal{B},\tilde{\mathcal{F}})$ interpolates between a fixed $(\Sigma_0,\mathcal{F}_0)$ and $(\Sigma,\mathcal{F})$.

For a general deformation of $(\Sigma,\mathcal{F})$

$$\delta\mathcal{W} = 0 \quad \Leftrightarrow \quad \text{F-flatness conditions } \mathcal{W}_m = 0$$

The same expression from the tension a DW given by a D-brane filling three flat directions and wrapping an internal generalized chain $(\mathcal{B},\tilde{\mathcal{F}})$ interpolating between $(\Sigma_0,\mathcal{F}_0)$ and $(\Sigma,\mathcal{F})$

$$\begin{aligned} T_{\rm DW} &= 2|\Delta\mathcal{W}| = \\ &= \int_{\mathcal{B}} P[\omega^{(DW)}]\wedge e^{\tilde{\mathcal{F}}} = \left|\int_{\mathcal{B}} P[e^{3A-\Phi}\hat{\Psi}_2]\wedge e^{\tilde{\mathcal{F}}}\right| .\end{aligned}$$

# Generalized forms on D-branes

The general infinitesimal deformation of $(\Sigma, \mathcal{F})$ is described by a section of the *generalized normal bundle*:

$$\mathcal{N}_{(\Sigma,\mathcal{F})} \equiv (T_M \oplus T_M^\star)|_\Sigma / T_{(\Sigma,\mathcal{F})} \; .$$

where $T_{(\Sigma,\mathcal{F})}$ is the generalized tangent bundle.

We can use $\mathcal{J}_2$ to split $\mathcal{N}_{(\Sigma,\mathcal{F})} \otimes \mathbb{C} = \mathcal{N}^{1,0}_{(\Sigma,\mathcal{F})} \oplus \mathcal{N}^{0,1}_{(\Sigma,\mathcal{F})}$ and one can define a differential

$$d_{(\Sigma,\mathcal{F})} : \Gamma(\Lambda^k \mathcal{N}^{0,1}_{(\Sigma,\mathcal{F})}) \to \Gamma(\Lambda^{k+1} \mathcal{N}^{0,1}_{(\Sigma,\mathcal{F})})$$

and the associated cohomology groups

$$H^k(\Sigma, \mathcal{F}) \equiv \ker\left(d_{(\Sigma,\mathcal{F})}|_k\right) / \mathrm{im}\left(d_{(\Sigma,\mathcal{F})}|_{k-1}\right)$$

From the DBI-action, it is possible to introduce a metric $G$ depending on $\hat{\Psi}_1$ on sections of $\Lambda^k \mathcal{N}^{0,1}_{(\Sigma,\mathcal{F})}$ and thus a codifferential

$$d^\dagger_{(\Sigma,\mathcal{F})} : \Gamma(\Lambda^k \mathcal{N}^{0,1}_{(\Sigma,\mathcal{F})}) \to \Gamma(\Lambda^{k-1} \mathcal{N}^{0,1}_{(\Sigma,\mathcal{F})})$$

# Deformations of calibrated generalized cycles

Consider an infinitesimal deformation given by $\mathbb{X}^{0,1} \in \Gamma(\mathcal{N}^{0,1}_{(\Sigma,\mathcal{F})})$. Then

- F-flatness ($(\Sigma, \mathcal{F})$ generalized complex):

$$d_{(\Sigma,\mathcal{F})}\mathbb{X}^{0,1} = 0$$

- Complexified D-flatness (including gauge-fixing):

$$d^{\dagger}_{(\Sigma,\mathcal{F})}\mathbb{X}^{0,1} = 0$$

Thus

$$\Delta_{(\Sigma,\mathcal{F})}\mathbb{X}^{0,1} = 0 \quad \text{where} \quad \Delta_{(\Sigma,\mathcal{F})} \equiv d_{(\Sigma,\mathcal{F})}d^{\dagger}_{(\Sigma,\mathcal{F})} + d^{\dagger}_{(\Sigma,\mathcal{F})}d_{(\Sigma,\mathcal{F})}.$$

The complexified D-flatness condition provides a gauge-fixing for the *$\mathcal{J}_2$-complexified* world-volume gauge transformations

$$\mathbb{X}^{0,1} \to \mathbb{X}^{0,1} + d_{(\Sigma,\mathcal{F})}\lambda\ . \tag{2}$$

Thus,

$$\text{massless fluctuations} = H^1(\Sigma, \mathcal{F}) \tag{3}$$

Consistent with BRST cohomology of topological branes in GC spaces, which is given by $H^{\bullet}(\Sigma, \mathcal{F})$ [Kapustin & Li, hep-th/0501071].

# A simple example: D3-brane on $\beta$-deformed complex manifold

Consider a complex manifold $M$ with holomorphic $(3,0)$ form $\Omega$. A $\beta$ deformation is given by $\beta \in H^0(\Lambda^2 T_M^{1,0})$, with $[\beta,\beta]=0$ and gives a type 1 pure spinor

$$e^{3A-\Phi}\hat{\Psi}_2 = \iota_\beta\Omega + \Omega \quad \Rightarrow \quad d\mathcal{W}_{\rm D3} = \iota_\beta\Omega \, .$$

Suppose to have a 0-cycle (D3-brane) at $z_0 \in M$. We have that

$$\text{F-flatness} \quad \Leftrightarrow \quad \iota_\beta\Omega|_{z_0} = 0 \quad \Leftrightarrow \quad \beta|_{z_0} = 0 \, .$$

Thus, the D3-brane must be located at a point were the *type jumps to* 3 ($\psi|_{z_0} = \Omega$). The differential complex is given by $\alpha \in \Lambda^k T_M^{1,0}|_{z_0}$, with differential acting as follows

$$d_{\{z_0\}}\alpha = -\partial\beta|_{z_0} \circ \alpha \equiv -\frac{1}{2(k-1)!}\partial_l\beta^{i_1 i_2}\alpha^{l i_3 \ldots i_{k+1}}\partial_{i_1}\wedge \ldots \wedge \partial_{i_{k+1}}|_{z_0} \, .$$

One thus obtains

$$H^1(\{z_0\}) = \{X \in T_M^{1,0}|_{z_0} : \partial\beta|_{z_0} \circ X = 0\} \, .$$

More directly, from the D3-brane superpotential,

$$\partial_i\partial_j\mathcal{W}_{\rm D3}|_{z_0} = \partial_i(e^{3A-\Phi}\hat{\Psi}_2)_j|_{z_0} = \frac{1}{2}(\partial_i\beta^{kl})\Omega_{klj}|_{z_0} \, .$$

# Future directions

- Global properties of the moduli space and higher order obstructions?
- How to extract the effective superpotential $\mathcal{W}_{\text{eff}}(\phi)$ for the massless fluctuations from the geometrical superpotential $\mathcal{W}(\Sigma, \mathcal{F})$?
- Coupling to closed string sector [Grana, Louis & Waldram '05; Benmachiche & Grimm '06]?
- Interesting nontrivial explicit realizations?
- ...

# Superpotentials from domain walls

Consider a BPS DW interpolating between two vacua $(\Sigma_1, \mathcal{F}_1)$ and $(\Sigma_2, \mathcal{F}_2)$. From field theory arguments [Cvetic et al., '91; Abraham & Townsend, '91] its tension is given by

$$T_{\rm DW} = 2|\Delta\mathcal{W}|\ .$$

In the D-brane realization, this DW is given by a D-brane filling three flat directions and wrapping an internal generalized chain $(\mathcal{B}, \tilde{\mathcal{F}})$ interpolating between $(\Sigma_1, \mathcal{F}_1)$ and $(\Sigma_2, \mathcal{F}_2)$.

$$\Downarrow$$

The tension is given by

$$T_{\rm DW} = \int_{\mathcal{B}} P[\omega^{(DW)}] \wedge e^{\tilde{\mathcal{F}}} = \Big|\int_{\mathcal{B}} P[e^{3A-\Phi}\hat{\Psi}_2)] \wedge e^{\tilde{\mathcal{F}}}\Big|\ .$$

$\Rightarrow$ The same expression for the superpotential is recovered!

# D-terms and Fayet-Iliopoulos terms

For a D$p$-brane wrapping an internal generalized cycle $(\Sigma, \mathcal{F})$, the D-term $\mathcal{D}$ has the explicit form

$$\mathcal{D} d^n\sigma \quad = \quad \mu_p P[e^{2A-\Phi}\mathrm{Im}\hat{\Psi}_1] \wedge e^{\mathcal{F}}|_{\mathrm{top}} \; .$$

Note that

$$\xi \equiv 2\pi\alpha' \int_\Sigma \mathcal{D} d^n\sigma$$

is constant under any continuous deformation of $(\Sigma, \mathcal{F})$

$$\Downarrow$$

The D-flatness condition $\mathcal{D} = 0$ can be satisfied only if $\xi = 0$.

*Natural interpretation:* $\xi$ is the FI term of the lowest KK gauge field, which has no charged chiral fields.

# FI terms and cosmic strings

We can obtain a D-brane cosmic string in the following way:

- Consider a D$\bar{\text{D}}p$-brane pair wrapping $(\Sigma, \mathcal{F})$ such that $\xi \neq 0$.
- By Sen's mechanism, a tachyonic vortex in the flat directions produces an effective string given by a D$(p-2)$-brane wrapping the same cycle $(\Sigma, \mathcal{F})$ and filling only two flat directions.

$$\Downarrow$$

The tension of a BPS cosmic string produced in this way is given by

$$T_{\text{string}} = \mu_{p-2} \int_{\Sigma} \omega^{(\text{string})} \wedge e^{\mathcal{F}} = 2\pi\xi \ .$$

Identical to the field-theory result of [Dvali, Kallosh & Van Proeyen, '03].

Further evidence that: *D-term strings* $\leftrightarrow$ *D-brane strings*

## Some superpotentials for D-branes on $SU(3)$-structure backgrounds

If the internal space has $SU(3)$ structure (i.e. $\eta^{(1)} = e^{i\varphi_1}\eta$ and $\eta^{(2)} = e^{i\varphi_2}\eta$), then

$$\hat{\Psi}^+ = -ie^{i(\varphi_1-\varphi_2)}e^{-iJ} \quad , \quad \hat{\Psi}^- = -e^{i(\varphi_1+\varphi_2)}\Omega \, .$$

- D5-brane

$$\mathcal{W} = \frac{1}{2}\int_{\mathcal{B}} P[e^{3A-\Phi}\Omega] \, ,$$

thus reproducing the superpotential proposed by [Witten, '96]

- D6-brane

$$\mathcal{W} = \int_{\mathcal{B}} \{P[J]\wedge\tilde{\mathcal{F}} + \frac{i}{2}P[J\wedge J] - \frac{i}{2}\tilde{\mathcal{F}}\wedge\tilde{\mathcal{F}}\}$$

- D7-brane

$$\mathcal{W}(\Sigma,\mathcal{F}) = \frac{1}{2}\int_{\mathcal{B}} P[e^{3A-\Phi}\Omega]\wedge\tilde{\mathcal{F}} \, .$$

See e.g. [Gomis, Marchesano & Mateos '05; Marchesano '06] for examples where these superpotentials generate flux-induced masses for the geometrical moduli.

# Probing the internal space with a D3-brane

- On more general IIB backgrounds, the integrable pure spinor has the form

$$\hat{\Psi}^- = \hat{\Psi}^-_{(1)} + \hat{\Psi}^-_{(3)} + \hat{\Psi}^-_{(5)} \qquad \text{with} \qquad \hat{\Psi}^-_{(5)} \sim \star_6 \hat{\Psi}^-_{(1)} \ .$$

  Thus,

$$\hat{\Psi}^-_{(1)}(y) = d\mathcal{W}_{\rm D3}(y)$$

  $\Rightarrow$ the D3-brane superpotential is trivial iff the internal space has $SU(3)$-structure!

- On $SU(3)$-structure backgrounds, one can still have a nontrivial D-term: $\mathcal{D}_{D3} = \cos(\varphi_1 - \varphi_2)$

  D-flatness condition $\mathcal{D}_{\rm D3} = 0 \Leftrightarrow$ the internal space is a warped Calabi-Yau of the kind discussed by [Graña-Polchinski].

# D7-brane on $SU(3)$ vacua and flux induced moduli lifting

$SU(3)$-structure IIB vacua $\Rightarrow$ The internal space is *complex* and $\exists$ holomorpic $(3,0)$ form $\Omega = e^{3A-\Phi}\hat{\Psi}^-$.

The D7-brane superpotential is given by

$$\mathcal{W}(\Sigma,\mathcal{F}) = \mathcal{W}_0 + \frac{1}{2}\int_{\mathcal{B}} P[\Omega] \wedge \tilde{\mathcal{F}} \,. \tag{4}$$

$\delta\mathcal{W}(\Sigma,\mathcal{F}) = 0 \quad \Leftrightarrow \quad \Sigma$ holomorphically embedded and $\mathcal{F}_{(2,0)} = 0$.

$h^{(2,0)}(\Sigma)$ possible massless chiral fields $t^i$ associated to the deformations of the holomorphic cycle $\Sigma$ generated by the $h^{(2,0)}(\Sigma)$ sections $X^i$ of $\mathcal{N}_\Sigma^{\rm hol}$.

We have $h^{(2,0)}(\Sigma)$ moduli lifting conditions [see also Gomis, Marchesano & Mateos, 0506179]:

$$\partial_i\mathcal{W} = \frac{1}{2}\int_\Sigma P_\Sigma[\imath_{X_i}\Omega] \wedge \mathcal{F} = 0$$

Furthermore, if $T_M^{1,0}|_\Sigma = T_\Sigma^{1,0} \oplus \mathcal{N}_\Sigma^{\rm hol}$ holomorphically, we have the $H$-induced masses

$$m_{ij}(t_0) \equiv (\partial_i\partial_j\mathcal{W})(t_0) = \frac{1}{2}\int_{\Sigma_0} P_{\Sigma_0}[\imath_{X_i}\Omega \wedge \imath_{X_j}H] \,. \tag{5}$$

If $\mathcal{C}$ is the configuration space of the generalized cycles $(\Sigma, \mathcal{F})$, it is possible to introduce an *almost* complex structure $\mathbb{J}$ on $\mathcal{C}$ such that

$$X \in T^{0,1}_{\mathcal{C}}|_{(\Sigma,\mathcal{F})} \quad \Rightarrow \quad X(\mathcal{W})|_{(\Sigma,\mathcal{F})} \equiv 0 \,. \tag{6}$$

Then, the superpotential $\mathcal{W}$ is 'holomorphic' with respect to $\mathbb{J}$.

It is also possible to introduce a formal symplectic structure $\Xi$ on $\mathcal{C}$ such that the deriving moment map $m(\Sigma, \mathcal{F})$ generating the world-volume gauge transformations coincides with the D-term $\mathcal{D}$.

The above almost complex and symplectic structures reproduce the known ones in the pure CY case.

*Like in that case, they are not trivially integrable and do not combine in a Kähler structure!*

Consider $T_M \oplus T_M^\star$ instead of $T_M$.

- Natural metric $\mathcal{I}(X+\xi, X+\xi) = \xi(X)$ of signature $(6,6)$
  $\Rightarrow$ Structure group $SO(6,6)$
- *Generalized almost complex structure*

$$\mathcal{J} : T_M \oplus T_M^\star \to T_M \oplus T_M^\star \quad ,$$

  such that $\mathcal{J}^2 = -1$ and $\mathcal{J}^T \mathcal{I} \mathcal{J} = \mathcal{I}$
  $\Rightarrow$ Reduction of the structure group to $U(3,3)$.
- $\Lambda^\bullet T^\star = \bigoplus_k \Lambda^k T^\star$ is the associated spinor bundle, where the action of $X + \xi \in T_M \oplus T_M^\star$ as "gamma matrix" is given by

$$(X+\xi)\cdot\omega = (\imath_X + \xi\wedge)\omega \quad \text{where} \quad \omega \in \Lambda^\bullet T^\star$$

- A spinor $\varphi \in \Lambda^\bullet T_M^\star$ is *pure* if it's null space $L_\varphi \subset T_M \oplus T_M^\star$ is of maximal dimension 6

- A globally defined pure spinor $\varphi \in C^\infty(\Lambda^\bullet T_M^\star \otimes \mathbb{C})$ such that $L_\varphi \cap \bar{L}_\varphi = 0$ (real index zero) defines a generalized almost complex structure $\mathcal{J}$, whose $+i$ eigenspace is given by $L_\varphi$ (reduction of the structure group to $SU(3,3)$).
- In $\mathcal{N} = 1$ backgrounds, $\Psi^+$ and $\Psi^-$ are pure spinors and their null spaces are of real index zero and have common three dimensional subspace
$\Rightarrow$ structure group further reduced to $SU(3) \times SU(3)$

$$\Downarrow$$

Our $\mathcal{N} = 1$ backgrounds have $SU(3) \times SU(3)$-structure group on $T_M \oplus T_M^\star$
$\Rightarrow$ This reduced structure group defines also the metric $g$ on $M$, since

$$G = -\mathcal{I}\mathcal{J}_+\mathcal{J}_- = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} . \qquad (7)$$

$G$ is a positive definite metric on $T_M \oplus T_M^\star$.

- Usual integrability condition for an almost complex structure $J : T_M \to T_M$, $J^2 = -1$ is that its $+i$ eigenspace is involutive:

$$Nij(X,Y) \simeq (1+iJ)[(1-iJ)X,(1-iJ)Y]_{Lie} = 0 \qquad (8)$$

- Analogous integrability condition for $\mathcal{J} : T_M \oplus T^\star_M \to T_M \oplus T^\star_M$, with $[.,.]_{Lie}$ substituted by the *(twisted) Courant bracket*

$$[X+\xi, Y+\eta]_{Courant} = [X,Y]_{Lie} + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(\imath_X\eta - \imath_T\xi) + \imath_Y\imath_X H\,. \qquad (9)$$

- If a pure spinor $\varphi$ is $d_H$-closed ($d_H = d + H\wedge$)
$\Rightarrow$ the associated generalized almost complex structure $\mathcal{J}$ is *integrable*

$$\Downarrow$$

The susy condition $d_H(e^{2A-\Phi}\Psi^\pm) = 0$ for IIA/IIB tells us that the internal space $M$ is a *generalized complex manifold*. Since $\Psi^\pm$ are globally defined $M$ is a *generalized Calabi-Yau structure* as defined by Hitchin.

- Complex case
  In this case $\varphi \propto \theta_1 \wedge \theta_2 \wedge \theta_3$ where $\theta_i, \bar{\theta}_i$ linearly independent ($\varphi \wedge \bar{\varphi} \neq 0$). Then

$$\mathcal{J} = \begin{pmatrix} -J & 0 \\ 0 & J^t \end{pmatrix} . \tag{10}$$

  The integrability condition $d_H \varphi = 0$ implies that
  - $J$ is an integrable complex structure,
  - $\varphi$ is a holomorphic $(3,0)$-form ($\mathcal{K}_M$ is trivial)
  - and $H^{(3,0)} = H^{(0,3)} = 0$.
- Symplectic case
  In this case $\varphi \propto e^{i\omega}$. In this case

$$\mathcal{J} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} . \tag{11}$$

  The condition $d_H \varphi = 0$ now implies that $d\omega = 0$ (symplectic) and $H = 0$

More generally one obtains a hybrid complex-symplectic structure which locally admits hybrid complex-symplectic coordinates (*generalized Darboux theorem* [Gualtieri]).

## $\mathcal{N} = 1$ vacua with $SU(3)$ structure: $\eta^{(1)} = a\eta$ and $\eta^{(2)} = b\eta$

- Introduce hermitian almost complex structure $J_{mn} = -i\eta_+^\dagger \hat{\gamma}_{mn}\eta_+$ and $(3,0)$ form $\Omega_{mnp} = -i\eta_-^\dagger \hat{\gamma}_{mnp}\eta_+$. These are such that

$$\frac{1}{3!}J \wedge J \wedge J = \frac{i}{8}\Omega \wedge \bar{\Omega} \quad , \quad J \wedge \Omega = 0$$

- Then the integrable spinors are:

$$\Psi^+ = \frac{a\bar{b}}{8}e^{-iJ} \quad (\text{IIA})\,, \quad \Psi^- = -\frac{iab}{8}\Omega \quad (\text{IIB}) \tag{12}$$

$$\Downarrow$$

For the $SU(3)$-structure subcase, the internal manifold $M$ is *symplectic* in type IIA and *complex* in type IIB

More generally the generalized complex structure of $M$ implies a local $2d + 4d$ splitting. For example, in the static $SU(2)$ case ($\eta^{(1)} \perp \eta^{(2)}$) one has

$$\Psi^+ \propto e^{i\omega_{(2d)}} \wedge \Omega_{(4d)}\,, \quad \text{IIA}$$
$$\Psi^- \propto e^{i\omega_{(4d)}} \wedge \Omega_{(2d)}\,, \quad \text{IIB}$$