Matching the Hagedorn Temperature in AdS/CFT

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Plan of the talk:

I Introduction

I Gauge theory side:

Hagedorn temperature for thermal N = 4 SYM on R £ S^3 with non-zero chemical potentials

Decoupling limit

Hagedorn temperature

I String theory side:

Decoupling limit of string theory on AdS_5 £ S^5

Penrose limit, matching of spectra

Computation and matching of the Hagedorn temperature

I Conclusions & Future directions

Introduction

- **AdS/CFT** correspondence $\longrightarrow$ a **strong/weak** coupling duality
- different route: compute the deconfinement temperature of planar N=4 SYM on RXS^3 with chemical potentials at weak coupling in a certain decoupling limit and match the result to the Hagedorn temperature of weakly coupled string theory on AdS_5XS^5 in the corresponding dual decoupling limit

- **Confinement/deconfinement** transition in N = 4 SYM on R £ S^3 believed to be dual to the **Hagedorn** transition in the dual string theory (Witten, Sundborg. Polyakov. Aharony et al.)
- planar N=4 SYM on RXS^3 at weak coupling and at large energies exhibits a **Hagedorn density of states $\rho(E)\sim E^{-1}\exp(T_H E)$**
- A succesful matching of the deconfinement temperature of the gauge theory should be done with the Hagedorn temperature of string theory on a **pp-wave** background

(Pando Zayas et al., Green et al., Sugawara, Brower et al., Grignani et al., Hyun et al, Bigazzi et al.)

- find a limit where the AdS/CFT correspondence is a **weak/weak** coupling duality

Gauge Theory side

Thermal partition function of U(N) N = 4 SYM on R X S^3 with chemical potentials

$$Z(\beta, \Omega_i) = \mathrm{Tr}_M \left(e^{-\beta D + \beta(\Omega_1 J_1 + \Omega_2 J_2 + \Omega_3 J_3)} \right)$$

$\beta = 1/T$: inverse temperature

D : Dilatation operator

J_i : R-charges for SU(4) R-symmetry of N = 4 SYM

Ω_i : Chemical potentials

M : The set of gauge invariant operators, given by linear combinations of all possible multi-trace operators $\mathrm{Tr}(\cdots)\mathrm{Tr}(\cdots)\cdots\mathrm{Tr}(\cdots)$

Introduce $x = e^{-\beta}$, $y_i = e^{\beta \Omega_i}$

Multi-trace partition function in the planar limit N=1 is given by

$$\log Z(x, y_i) = -\sum_{k=1}^{\infty} \log \left[1 - z(\omega^{k+1} x^k, y_i^k) \right]$$

z: letter partition function

$\omega = e^{2\pi i}$ is -1 when uplifted to half-integer

Free thermal N = 4 SYM on R £ S^3

$D = D_0$ The dilatation operator is just the bare scaling dimension

The partition function for free planar N = 4 SYM on R £ S^3 is

$$\log Z(x, y_i) = -\sum_{k=1}^{\infty} \log\left[1 - z(\omega^{k+1} x^k, y_i^k)\right]$$

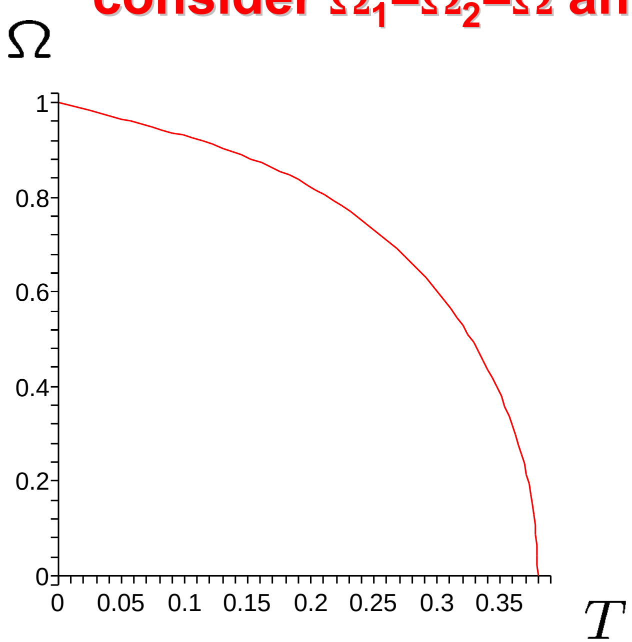
Sundborg. Polyakov. Aharony et al.
Yamada and Yaffe. MO and T. Harmark

$Z(x,y_i)$ has a singularity when $z(x,y_i) = 1$! **The Hagedorn singularity**

define the Hagedorn temperature $T_H(\Omega_1,\Omega_2,\Omega_3)$

consider $\Omega_1=\Omega_2=\Omega$ and $\Omega_3=0$

$$\Omega \to 1 : T_H(\Omega) \simeq \frac{1-\Omega}{\log 2}$$



we are interested in the limit:

$$T \to 0,\ \ \Omega \to 1,\ \ \tilde{T} \equiv \frac{T}{1-\Omega}\ \text{fixed}$$

The Hagedorn temperature in the limit is:

$$\tilde{T}_H = \frac{1}{\log 2}$$

→ same as in the **SU(2)** sector

Interacting N = 4 SYM on R £ S^3:

consider weakly coupled U(N) N = 4 SYM on R £ S^3

and $\Omega_1=\Omega_2=\Omega$ and $\Omega_3=0$

$$\lambda = \frac{g_{\mathrm{YM}}^2 N}{4\pi^2}$$

Decoupling limit

$$T \to 0,\ \Omega \to 1,\ \lambda \to 0,\ \tilde{T} \equiv \frac{T}{1-\Omega} \text{ fixed},\ \tilde{\lambda} \equiv \frac{\lambda}{1-\Omega} \text{ fixed},\ N \text{ fixed}$$

Full partition function: $Z(\beta,\Omega) = \mathrm{Tr}_M \left(e^{-\beta D + \beta\Omega J}\right)$ $\longrightarrow$ $J = J_1 + J_2$

Dilatation operator: $D = D_0 + \lambda D_2 + \lambda^2 D_4 + \cdots$

(E.g. Beisert's thesis)

Consider the weight factor:

$$e^{-\beta D + \beta\Omega J} = \exp\left(-\beta(D_0 - J) - \beta(1-\Omega)J - \beta\lambda D_2 - \beta\sum_{n=2}^{\infty}\lambda^n D_{2n}\right)$$

Since $\beta \to 1$, only the states with D – J being of order 1 – Ω survive
! Effective truncation to states with D_0 = J ! The SU(2) sector

Decoupling limit

$$T \to 0,\ \Omega \to 1,\ \lambda \to 0,\ \tilde{T} \equiv \frac{T}{1-\Omega} \text{ fixed},\ \tilde{\lambda} \equiv \frac{\lambda}{1-\Omega} \text{ fixed},\ N \text{ fixed}$$

From the analysis before we see that the full partition function in this limit is

$$Z(\tilde{\beta}) = \mathrm{Tr}_{\mathcal{H}}\left(e^{-\tilde{\beta}H}\right) \qquad \text{with} \quad \tilde{\beta} \equiv \beta(1-\Omega)$$

SU(2) sector

Hilbert space: $\mathcal{H} = \{\alpha \in M | (D_0 - J)\alpha = 0\}$

Hamiltonian: $H = D_0 + \tilde{\lambda} D_2$

The Hamiltonian truncates
! has only the bare + one-loop term

Note also: $\tilde{\lambda}$ can be finite, i.e. it does not have to be small

➡ **N = 4 SYM is weakly coupled in this limit**

Planar limit N = 1 ! we can focus on the single-trace sector

$$\mathrm{Tr}(XZZX\cdots Z) \longrightarrow |\downarrow\uparrow\uparrow\downarrow\cdots\uparrow\rangle$$ spin chain

$Z:\uparrow,\ X:\downarrow$

$$D_2 = \frac{1}{2}\sum_{i=1}^{L}(I_{i,i+1} - P_{i,i+1})$$

$\tilde{\lambda} D_2$: Hamiltonian of ferromagnetic $XXX_{1/2}$ Heisenberg spin chain (Minahan & Zarembo)

L is the length of single-trace operator / spin chain

Total Hamiltonian: $H = D_0 + \tilde{\lambda} D_2 = L + \tilde{\lambda} D_2$

In the limit $T \to 0,\ \Omega \to 1\,,\ \lambda \to 0\,,\ \tilde{T} \equiv \dfrac{T}{1-\Omega}$ fixed, $\tilde{\lambda} \equiv \dfrac{\lambda}{1-\Omega}$ fixed

planar N = 4 SYM on R £ S[3] has the partition function

$$\log Z(\tilde{\beta}) = \sum_{n=1}^{\infty}\sum_{L=1}^{\infty}\frac{1}{n}e^{-nL\tilde{\beta}}Z_L^{(\mathrm{XXX})}(n\tilde{\beta})$$

$$\tilde{\beta} \equiv \beta(1-\Omega)$$

$$Z_L^{(XXX)}(\tilde{\beta}) = \mathrm{Tr}_L\left(e^{-\tilde{\beta}\tilde{\lambda}D_2}\right) \longleftarrow$$

Partition function for the ferromagnetic $XXX_{1/2}$ Heisenberg spin chain

The ferromagnetic Heisenberg model is obtained as a limit of weakly coupled planar N = 4 SYM

Spectrum of gauge theory from Heisenberg chain

The Hamiltonian of planar N=4 SYM in the decoupling limit is

$$\rightarrow \quad H = D_0 + \tilde{\lambda} D_2 = L + \tilde{\lambda} D_2$$

we want to find the spectrum of $\tilde{\lambda} D_2$

Define the total spin S_z and the the lenght L of the single trace operator:

$$S_z = \frac{J_1 - J_2}{2} \qquad\qquad L = J_1 + J_2$$

Spectrum $\longrightarrow$ Vacua (**D_2 = 0**) plus excitations (**magnons**)

Vacua are given by: $D_2 = 0$

Exists a vacuum for each value of S_z:

$$|S_z\rangle_L \sim \text{Tr}\left(\text{sym}(Z^{J_1} X^{J_2})\right) \qquad \text{where} \quad \begin{cases} J_1 = \frac{1}{2} L + S_z \\ J_2 = \frac{1}{2} L - S_z \end{cases}$$

These L+1 states are all the possible states for which D_2 = 0, i.e. all the possible vacua

The vacua $|S_z\rangle_L$ are the chiral primaries of N = 4 SYM obeying $D_0 = J_1 + J_2$ (=L) ! The low energy excitations are ’close’ to BPS

Low energy excitations: Magnons

In the thermodynamic limit (large L) the spectrum of the magnons, determined using Bethe ansatz technique + integrability, is

$$E = \frac{2\pi^2\tilde{\lambda}}{L^2}\sum_{n\neq 0} n^2 M_n \ , \quad \sum_{n\neq 0} n M_n = 0$$

Hagedorn temperature from Heisenberg chain:

Consider the partition function

$$\log Z(\tilde{\beta}) = \sum_{n=1}^{\infty}\sum_{L=1}^{\infty}\frac{1}{n}e^{-nL\tilde{\beta}}\mathrm{Tr}_L\left(e^{-n\tilde{\beta}\tilde{\lambda}D_2}\right)$$

Define $V(t) \equiv \lim_{L\to\infty}\frac{1}{L}\log \mathrm{Tr}_L\left(e^{-t^{-1}D_2}\right)$

Notice: $f(t) = -tV(t)$ ⟵ f(t) is the thermodynamic limit of the free energy per site for the Heisenberg chain

so we have that

$$e^{-nL\tilde{\beta}}\mathrm{Tr}_L\left(e^{-n\tilde{\beta}\tilde{\lambda}D_2}\right) \simeq \exp\left(-nL\tilde{\beta} + LV((n\tilde{\beta}\tilde{\lambda})^{-1})\right) \quad \text{for} \quad L\to\infty$$

n=1 gives the first singularity

the Hagedorn singularity is given by:

$$\tilde{T}_H = \left[V\left(\tilde{\lambda}^{-1}\tilde{T}_H\right)\right]^{-1}$$

general relation between thermodynamics of Heisenberg chain and Hagedorn temperature

$$\tilde{T}_H = \frac{1}{V\left(\tilde{\lambda}^{-1}\tilde{T}_H\right)} \longleftarrow$$ Defines $\tilde{T}_H$ as function of $\tilde{\lambda}$

$\tilde{\lambda} \ll 1 \longleftrightarrow$ High temperatures
t À 1

$\tilde{\lambda} \gg 1 \longleftrightarrow$ Low temperatures
t ¿ 1

Large $\tilde{\lambda}$/low temperatures:

Using the low-energy spectrum $E = \frac{2\pi^2\tilde{\lambda}}{L^2}\sum_{n\neq 0} n^2 M_n \; , \quad \sum_{n\neq 0} n M_n = 0$

we find for t ¿ 1: $V(t) = \zeta\left(\frac{3}{2}\right)\sqrt{\frac{t}{2\pi}}$

This gives ➡

$$\tilde{T}_H = (2\pi)^{1/3}\left[\zeta\left(\frac{3}{2}\right)\right]^{-2/3}\tilde{\lambda}^{1/3} \quad \text{for} \quad \tilde{\lambda} \gg 1$$

Hagedorn temperature of weakly coupled planar N = 4 SYM on R £ S[3] in the limit

$$T \to 0,\ \Omega \to 1\,,\ \lambda \to 0\,,\ \tilde{T} \equiv \frac{T}{1-\Omega}\ \text{fixed},\ \tilde{\lambda} \equiv \frac{\lambda}{1-\Omega}\ \text{fixed}$$

Correction computed in the Heisenberg chain, gives the following correction

(Takahashi)

$$\tilde{T}_H = \frac{(2\pi)^{1/3}}{\zeta(\frac{3}{2})^{2/3}}\tilde{\lambda}^{1/3} + \frac{4\pi}{3\zeta(\frac{3}{2})^2} + \mathcal{O}(\tilde{\lambda}^{-1/3})$$

String Theory side

Consider type IIB superstring theory in the pp-wave background:

$$ds^2 = -4dx^+dx^- - \mu^2 \sum_{I=3}^{8} x^I x^I (dx^+)^2 + \sum_{i=1}^{8} dx^i dx^i + 4\mu x^2 dx^1 dx^+$$ ← Michelson

$$F_{(5)} = 2\mu dx^+ (dx^1 dx^2 dx^3 dx^4 + dx^5 dx^6 dx^7 dx^8)$$

x^1 a flat direction

Penrose limit: Bertolini, de Boer, Harmark, Imeroni & Obers

with currents: $H_{\text{lc}} = i\partial_{x^+} = \mu(E-J)\ ,\ \ p^+ = \frac{i}{2}\partial_{x^-} = \frac{E+J}{2\mu R^2}\ ,\ \ p_1 = -i\partial_{x^1} = \frac{2S_z}{R}$

$J_1 = \frac{1}{2}J + S_z\ ,\ \ J_2 = \frac{1}{2}J - S_z$ angular momenta of the strings on the 3-sphere

Light-cone string spectrum:

$$l_s^2 p^+ H_{\text{lc}} = 2fN_0 + \sum_{n\neq 0}\left[(\omega_n + f)N_n + (\omega_n - f)M_n\right] + \sum_{n\in Z}\sum_{I=3}^{8} \omega_n N_n^{(I)}$$
$$+ \sum_{n\in Z}\left[\sum_{b=1}^{4}\left(\omega_n - \frac{1}{2}f\right)F_n^{(b)} + \sum_{b=5}^{8}\left(\omega_n + \frac{1}{2}f\right)F_n^{(b)}\right]$$

with $f = \mu l_s^2 p^+\ ,\ \ \omega_n = \sqrt{n^2 + f^2}$

Level-matching condition: $\sum_{n\neq 0} n\left[N_n + M_n + \sum_{I=3}^{8} N_n^{(I)} + \sum_{b=1}^{8} F_n^{(b)}\right] = 0$

! The pp-wave has the right vacuum structure due to the flat direction

$H_{lc} = 0$ \$ $E = J$ A vacuum for each p_1 \$ a vacuum for each S_z

the limit is

$$\mu \to \infty, \quad T \to 0, \quad \Omega \to 1, \quad \tilde{T} \equiv \frac{T}{1-\Omega} \text{ fixed}, \ l_s, p^+ \text{ fixed}$$

$$\tilde{H} = \frac{H_{lc}}{\sqrt{1-\Omega}} \text{ fixed}, \quad \tilde{g}_s = \frac{g_s}{1-\Omega} \text{ fixed}, \quad \tilde{\mu} = \mu\sqrt{1-\Omega} \text{ fixed}$$

take the limit on the spectrum (only the bosonic part)

$$l_s^2 p^+ \tilde{H}_{lc} = 2fN_0 + \sum_{n\neq 0} [(\omega_n + f)N_n + (\omega_n - f)M_n] + \sum_{n\in Z}\sum_{I=3}^{8} \omega_n N_n^{(I)}$$

For $f = \mu l_s^2 p^+ \to \infty$ ➡ $\omega_n \sim f + \frac{n^2}{2f}$

l Only the modes with number operator M_n survive

$$l_s^2 p^+ \tilde{H}_{lc} = \sum_{n\neq 0} (\omega_n - f)M_n \sim \sum_{n\neq 0} \frac{n^2}{2f} M_n$$

l Presence of flat direction gives non-trivial spectrum after limit,

in terms of gauge theory quantities

$$\frac{\tilde{H}_{lc}}{\tilde{\mu}} = \frac{2\pi^2\tilde{\lambda}}{J^2} \sum_{n\neq 0} n^2 M_n \ , \quad \sum_{n\neq 0} nM_n = 0$$

Matches spectrum of weakly coupled gauge theory!

Computation of Hagedorn temperature

1) Using the spectrum after the decoupling limit

$$\tilde{H}_{\text{lc}} \sim \sum_{n\neq 0} \frac{n^2}{2l_s^2 p^+ f} M_n$$

the partition function has singularity for $\tilde{\beta}_H^{3/2} = \zeta(3/2)\sqrt{\frac{2\pi}{\tilde{\lambda}}}$

we get

$$\tilde{T}_H = (2\pi)^{1/3}\left[\zeta\left(\frac{3}{2}\right)\right]^{-2/3}\tilde{\lambda}^{1/3}$$

Matches the Hagedorn temperature of gauge theory / Heisenberg chain

2) Using the full spectrum before the limit. The eq. for the Hagedorn is

$$\tilde{\beta}_H \pi\sqrt{\tilde{\lambda}} = 2\sum_{p=1}^{\infty}\frac{1}{p}\left[3+\cosh\left(\frac{\tilde{\beta}_H p}{1-\Omega}\right) - 4(-1)^p\cosh\left(\frac{\tilde{\beta}_H p}{2(1-\Omega)}\right)\right] K_1\left(\frac{\tilde{\beta}_H p}{1-\Omega}\right)$$

Sugawara

taking now the limit we get again $\tilde{T}_H = (2\pi)^{1/3}\left[\zeta\left(\frac{3}{2}\right)\right]^{-2/3}\tilde{\lambda}^{1/3}$

Check on the validity of the decoupling limit

→ verifies commutativity of limits

Conclusions...

l A solvable sector of AdS/CFT

Ferromagnetic Heisenberg spin chain	⟷	Limit of weakly coupled planar N = 4 SYM	⟷	Limit of free strings on AdS_5 £ S^5

A spin chain/gauge theory/string theory correspondence

l Explicit matching for large $\tilde{\lambda}$/L spectrum and Hagedorn temperature

$$E = \frac{2\pi^2\tilde{\lambda}}{L^2}\sum_{n\neq 0} n^2 M_n \ , \quad \sum_{n\neq 0} n M_n = 0 \qquad \tilde{T}_H = (2\pi)^{1/3}\left[\zeta\left(\frac{3}{2}\right)\right]^{-2/3}\tilde{\lambda}^{1/3}$$

...& Future directions

l Heisenberg chain with magnetic field, (work in progress with T.Harmark and K.R.Kristjánsson)

l λ corrections to spectrum/thermodynamics

l $\tilde{\lambda}^{-1/3}$ corrections on the string side

l SU(2|3) sector: Which pp-wave?