Gravitational theories coupled to matter as invariant theories under Kac-Moody algebras

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M-Theory

- Candidate for the unification of all fundamental interactions
- M-theory would encompass all superstring theories and in particular limit at low energy the eleven dimensional supergravity

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$$\downarrow$$

Study of hidden symmetries (= exhibited by dimensional reduction) would allow a best understanding of the structure of this unified theory

$$S = \frac{1}{8\pi G^{(D)}} \int d^D x \sqrt{-g} \left(R - \frac{1}{2} \sum_{u=1}^{q} \partial_M \Phi^u \partial^M \Phi^u - \sum_n \frac{1}{2n!} e^{\sum_u a_n^u \Phi^u} F_{(n)}^2 \right)$$

- gravity : $g_{\mu\nu}$
- dilatons : Φ^u
- matter fields : $F_{(n)} = dA_{(n-1)}$

Original formulation of gravitational theories coupled to matter fields and dilatons in terms of actions invariant under Kac-Moody algebras

$\longrightarrow$ study of hidden symmetries

Outline

1. Dimensional reduction and coset symmetries
   - Compactification down to 3 dimensions and coset Lagrangian $\mathcal{G}/\mathcal{K}$
   - Beyond 3 dimensions : Kac-Moody algebras
2. $\mathcal{G}^{+++}$- invariant action
   - Construction of $\mathcal{G}^{+++}$- invariant action : $\mathcal{S}_{\mathcal{G}^{+++}}$
   - E_8^{+++} - invariant action and link with the 11- dimensional supergravity
3. $\mathcal{G}^{++}$-invariant actions
   - $\mathcal{G}_C^{++}$ and cosmological solutions
   - $\mathcal{G}_B^{++}$ and branes solutions
4. Weyl transformations and their consequences
   - Signatures
5. Conclusions and perspectives

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Compactification down to 3 dimensions

D dimensions : $\mathcal{L} = \sqrt{-g}\left(R - \frac{1}{2}\sum_{u=1}^{q} \partial_M \Phi^u \partial^M \Phi^u - \sum_n \frac{1}{2n!} e^{\sum_u a_n^u \Phi^u} F_{(n)}^2\right)$

↓ *Compactification on a torus* T^{D-3}

3 dimensions : $\mathcal{L}_{3D} = \sqrt{-g}\left(R - \frac{1}{2}\,\partial_\mu \bar{\varphi} \cdot \partial^\mu \bar{\varphi} - \frac{1}{2}\sum_{\bar{\alpha}} e^{\sqrt{2}\,\bar{\alpha}\cdot\bar{\varphi}}\,\partial_\mu \chi_{\bar{\alpha}}\,\partial^\mu \chi_{\bar{\alpha}}\right)$

The 3-dimensional Lagrangian contain only scalars coupled to gravity.

Expected symmetry = $GL(D-3,\mathbb{R})$

BUT !

Some theories exhibit a much larger symmetry

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Scalar part of $\mathcal{L}_{3D}$ $\simeq$ Coset Lagrangian $\mathcal{L}_{\mathcal{G}/\mathcal{K}}$

(invariant under transformations $\mathcal{G}/\mathcal{K}$)

$\mathcal{G} \longrightarrow$ simple Lie group

$\mathcal{K} \longrightarrow$ maximal compact subgroup of $\mathcal{G}$

Cremmer, Julia '79
Marcus, Schwarz '83

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Scalar part of $\mathcal{L}_{3D}$ $\simeq$ Coset Lagrangian $\mathcal{L}_{\mathcal{G}/\mathcal{K}}$

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necessary condition

$\bar{\alpha}$ identified

to positive roots of $\mathcal{G}$

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Example : 11-dimensional supergravity

$$\mathcal{L} = \sqrt{-g}\left(R(g_{\mu\nu}) - \frac{1}{2.4!}\ F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} +\ C.S.\ \right)$$

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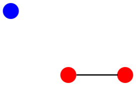

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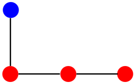

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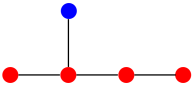

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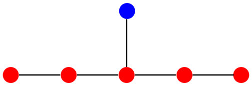
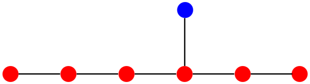


Let us reduce the dimensions down to $D = \; 5$

$$\mathcal{L} = \sqrt{-g}\left(R(g_{\mu\nu}) - \frac{1}{2.4!}\ F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} +\ C.S.\ \right)$$



Let us reduce the dimensions down to $D =\ 4$

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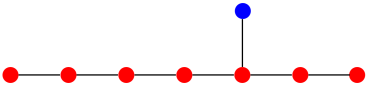

Dynkin diagram of E_8

Let us reduce the dimensions down to $D =\ 3$

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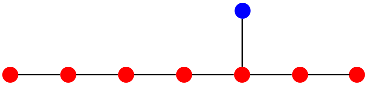

Dynkin diagram of E_8

- Red vertices define the gravity line. It represents simple roots related to fields coming from $g_{\mu\nu}$.
- The blue vertex is related to field resulting from $F_{\mu\nu\rho\sigma}$.

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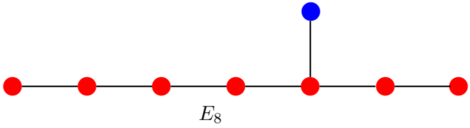
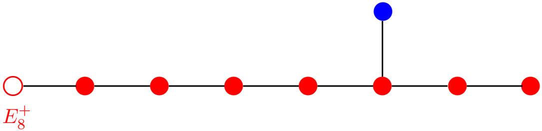

E_8

Beyond 3 dimensions : Kac-Moody algebras



- It has been showed that the reduced theory to 2 dimensions is connected to an infinite dimensional symmetry $\mathcal{G}^+$ (affine extension of $\mathcal{G}$).

Nicolai '87

- $D = 1 \Longrightarrow \mathcal{G}^{++}$ (Overextension of $\mathcal{G}$).

Julia '82

- $D = 0 \Longrightarrow \mathcal{G}^{+++}$ (Very-extension of $\mathcal{G}$).

West '01
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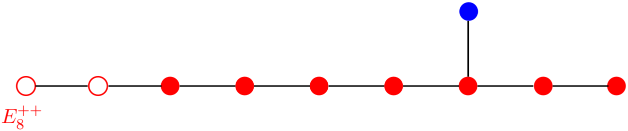

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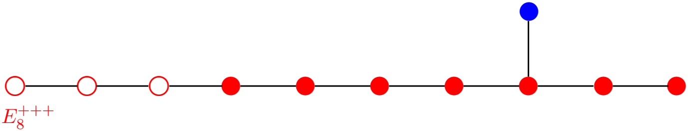

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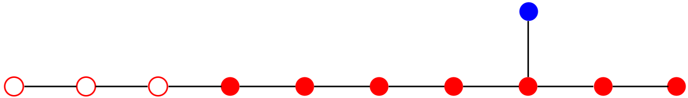

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$\mathcal{G}^+$, $\mathcal{G}^{++}$ and $\mathcal{G}^{+++}$ are Kac-Moody algebras
=
Infinite dimensional Lie algebras

Other examples

- Pure gravity in D dimensions $\rightarrow$ A_{D-3}^{+++}
- Effective action of closed bosonic string in 26 dimensions $\rightarrow$ D_{24}^{+++}

All simple maximally non-compact Lie group $\mathcal{G}$ could be generated from the reduction down to 3 dimensions of suitably chosen actions.

Cremmer, Julia, Lu, Pope '99

It was conjectured that these actions possess the very-extended Kac-Moody symmetries $\mathcal{G}^{+++}$.

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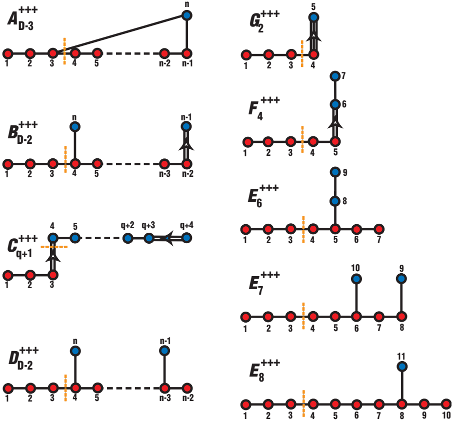

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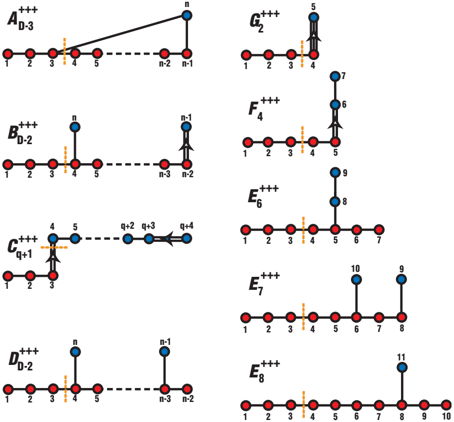

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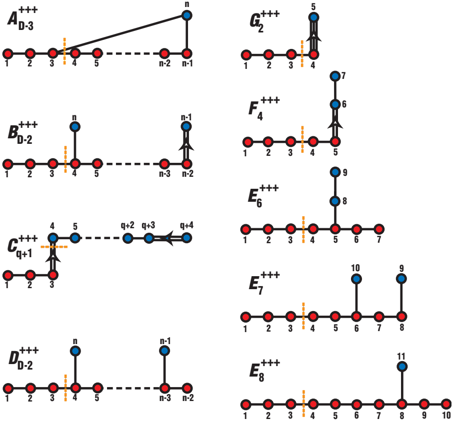

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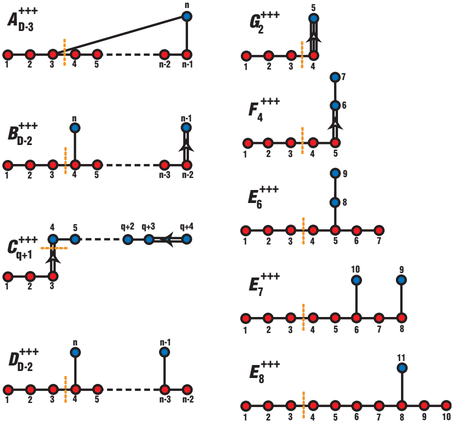

Possible existence of this Kac-Moody symmetry

motivates

Construction of action explicitly invariant under $\mathcal{G}^{+++}$

Englert, Houart '03

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Construction of $\mathcal{G}^{+++}$- invariant action : $\mathcal{S}_{\mathcal{G}^{+++}}$

$\mathcal{S}_{\mathcal{G}^{+++}}$ is defined in terms of an infinity of fields $\phi(\xi)$ belonging to a coset $\mathcal{G}^{+++}/\mathcal{K}^{+++}$.

- ξ is a world-line parameter.
- $\mathcal{K}^{+++}$ is the subalgebra invariant under a *temporal involution* Ω_1 which ensures that the action is $SO(1, D-1)$ invariant.

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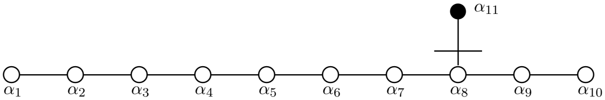
Parametrization of the Coset $\mathcal{G}^{+++}/\mathcal{K}^{+++}$

$$\mathcal{V}(\xi) = exp^{\mathcal{B}^a \phi_a(\xi)}$$

- $\mathcal{B}^a$ = generators of $\mathcal{G}^{+++}$ belonging to the Borel subalgebra (Cartan + positive root generators)
- To each $\mathcal{B}^a$, one associates a 'field' $\phi_a(\xi)$

Example : level decomposition of E_8^{+++}

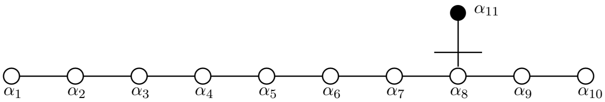
A level decomposition of E_8^{+++} is performed with respect to the sub-algebra A_{10} of its gravity line.



$$\alpha = l\alpha_{11} + \sum_{i=1}^{10} a_i \alpha_i$$

The level l counts the number of times the simple root α_{11} (not contained in the gravity line) appears in A_{10} irreducible representation.

Nicolai, Fischbacher '03



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Level l	*Generators*	*Fields*
0	$K^a{}_b \to GL(11)$	$h_a{}^b(\xi)$
1	$R^{a_1\, a_2\, a_3}$	$A_{a_1\, a_2\, a_3}(\xi)$
2	$R^{a_1\, a_2\, a_3\, a_4\, a_5\, a_6}$	$A_{a_1\, a_2\, a_3\, a_4\, a_5\, a_6}(\xi)$
3	$R^{a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, a_7\, a_8,\, b}$	$A_{a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, a_7\, a_8,\, b}(\xi)$
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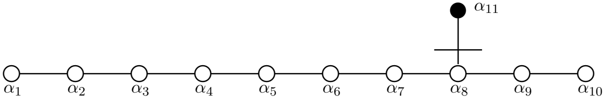

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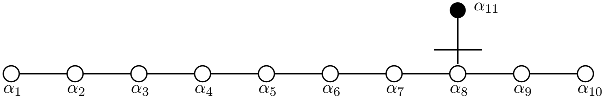

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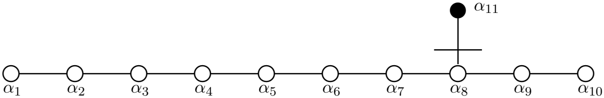
Level decomposition of E_8^{+++}



$$\alpha = l\alpha_{11} + \sum_{i=1}^{10} a_i \alpha_i$$

Level l	*Generators*	*Fields*
0	$K^a{}_b \to GL(11)$	$h_a{}^b(\xi)$
1	$R^{a_1\, a_2\, a_3}$	$A_{a_1\, a_2\, a_3}(\xi)$
2	$R^{a_1\, a_2\, a_3\, a_4\, a_5\, a_6}$	$A_{a_1\, a_2\, a_3\, a_4\, a_5\, a_6}(\xi)$
3	$R^{a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, a_7\, a_8,\, b}$	$A_{a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, a_7\, a_8,\, b}(\xi)$
...	...	...

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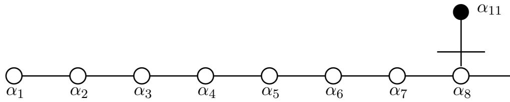

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...	...	...

Construction of $\mathcal{S}_{\mathcal{G}^{+++}}$

$$\mathcal{V}(\xi) = exp\Big(\underbrace{\sum_{a\geq b} h_b{}^a(\xi) K^b{}_a}_{Level\ 0}\Big) exp\Big(\underbrace{\sum A_{a_1\,a_2\,a_3}(\xi)\, R^{a_1\,a_2\,a_3} + \dots}_{Level\ \geq 1}\Big)$$

Defining

$$\frac{dv(\xi)}{d\xi} = \frac{d\mathcal{V}}{d\xi}\mathcal{V}^{-1} \qquad \frac{d\tilde{v}(\xi)}{d\xi} = -\Omega_1\,\frac{dv(\xi)}{d\xi} = \tilde{\mathcal{V}}^{-1}\frac{d\tilde{\mathcal{V}}}{d\xi} \qquad \mathcal{P} = \frac{1}{2}\left(\frac{dv}{d\xi} + \frac{d\tilde{v}}{d\xi}\right)$$

Ω_1 = temporal involution allows identification of index 1 to a time coordinate

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E_8^{+++} - INVARIANT ACTION AND LINK WITH THE 11-DIMENSIONAL SUPERGRAVITY

Action invariant under $\mathcal{G}^{+++}$

$\mathcal{S}_{E_8^{+++}}$

11-dimensional supergravity

E_8^{+++} - INVARIANT ACTION AND LINK WITH THE 11-DIMENSIONAL SUPERGRAVITY

Action invariant under $\mathcal{G}^{+++}$ $\overset{?}{\leftrightsquigarrow}$ 11-dimensional supergravity

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★ How to interpret the parameter ξ ?

★ How to interpret the 'fields' $\phi(\xi)$?

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$\downarrow$

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Study of $\mathcal{G}^{++}$

$\mathcal{G}^{++}$-invariant actions

The formulation of space-time theories as invariant theories under Kac-Moody algebras, includes for all $\mathcal{G}^{+++}$, 2 inequivalent invariant actions under $\mathcal{G}^{++} \subset \mathcal{G}^{+++}$:

- $\mathcal{G}_C^{++}$
- $\mathcal{G}_B^{++}$

Englert, Henneaux, Houart $'04$

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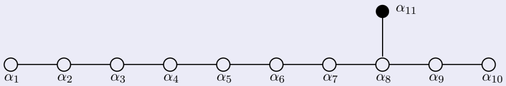

Englert, Henneaux, Houart '04

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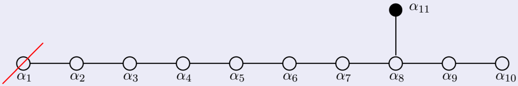

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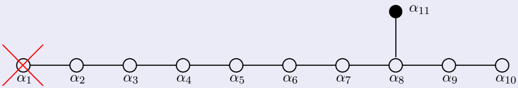

Englert, Henneaux, Houart '04

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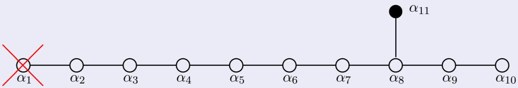

Construction of actions $\mathcal{S}_{\mathcal{G}^{++}}$ from $\mathcal{S}_{\mathcal{G}^{+++}}$ by a consistent truncation :

- truncation : one puts to zero all the fields multiplying generators involving the deleted root α_1
- solutions of EOM $\mathcal{S}_{\mathcal{G}^{++}}$ = solutions of EOM $\mathcal{S}_{\mathcal{G}^{+++}}$

Englert, Henneaux, Houart '04

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$\mathcal{G}_C^{++}$- INVARIANT ACTION

Cosmological solutions in the vicinity of space-like singularity

reveals

$\mathcal{G}_C^{++}$

Link with space-time theories

The restriction of $\mathcal{S}_{\mathcal{G}_C^{++}}$ to a definite number of lowest levels is equal to the corresponding space-time theory in which the fields depend only on the time coordinate

↓

ξ = time coordinate

and euclidiean signature in $D-1$ dimensions ($\hat{\mu} = 2, \ldots, D$)

	Fields belonging to $\mathcal{S}_{E_8^{++}}$		*Fields of supergravity depending on time*
level 0	$g_{\hat{\mu}\hat{\nu}}(t) = fct(h_a{}^b)$	$\leftrightsquigarrow$	metric
level 1	$A_{\hat{\mu}\hat{\nu}\hat{\rho}}(t)$	$\leftrightsquigarrow$	3-form electric potential
level 2	$A_{\hat{\mu}_1\ldots\hat{\mu}_6}(t)$	$\leftrightsquigarrow$	6-form magnetic potential (dual of the 3-form)
level 3	$A_{\hat{\mu}_1\ldots\hat{\mu}_8,\hat{\nu}}(t)$	$\leftrightsquigarrow$	'dual' of the metric

Damour, Henneaux, Nicolai '02

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Same construction as $\mathcal{G}_C^{++}$ but after a Weyl reflection W_{α_1} :

Weyl reflection is an automorphism of $\mathcal{G}^{+++}$ which transform a root of $\mathcal{G}^{+++}$ to another root.

$\mathcal{G}_B^{++}$- INVARIANT ACTION : CONSTRUCTION

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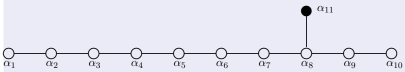

1) Weyl reflection W_{α_1}

Exchange of the identification of the coordinates 1 and 2 :

- 1(temporal) $\rightarrow$ 1 (spatial)
- 2(spatial) $\rightarrow$ 2 (temporal)
- others unchanged (spatial)

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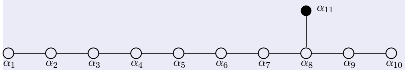

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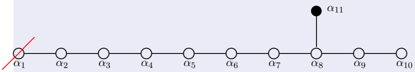

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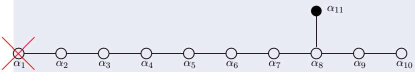

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$\mathcal{G}_B^{++}$- INVARIANT ACTION

Englert, Houart '03,' 04

Link with space-time theories

$\mathcal{G}_B^{++}$ ⟶ Solutions are identical to the ones of covariant Einstein and fields equations describing intersecting extremal branes smeared in all dimensions but one

↓

ξ = spatial non-compact coordinate
and Lorentz signature in $D-1$ dimensions ($\hat{\mu} = 2, \ldots, D$)

	Fields belonging to $\mathcal{S}_{E_8^{++}}$		*Branes of M-theory*
level 0	$g_{\hat{\mu}\hat{\nu}}(x)$	⇝	KK-wave (0-brane)
level 1	$A_{\hat{\mu}\hat{\nu}\hat{\rho}}(x) + g_{\hat{\mu}\hat{\nu}}(x)$	⇝	$M2$ (2-brane)
level 2	$A_{\hat{\mu}_1\ldots\hat{\mu}_6}(x) + g_{\hat{\mu}\hat{\nu}}(x)$	⇝	$M5$ (5-brane)
level 3	$A_{\hat{\mu}_1\ldots\hat{\mu}_8,\hat{\nu}}(x) + g_{\hat{\mu}\hat{\nu}}(x)$	⇝	$KK6$-monopole

★ Intersection rules neatly encoded in $\mathcal{G}_B^{++} \Longrightarrow$ orthogonality condition

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Non-Commutativity of the temporal involution with Weyl reflection

A Weyl transformation on a generator T of $\mathcal{G}^{+++}$ can be expressed as a conjugaison by a group element U_W of $\mathcal{G}^{+++}$.

$$T \longrightarrow U_W T U_W^{-1}$$

Different Lorentz signatures (t, s) can be obtained by Weyl transformations.

Why ?

because of the non-commutativity of Weyl transformation with the temporal involution Ω :

$$U_W \Omega T U_W^{-1} = \Omega' U_W T U_W^{-1}$$

Englert, Henneaux, Houart $'04$

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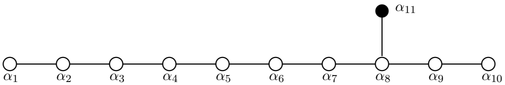
Englert, Henneaux, Houart '04

Signatures of E_8^{+++}



Weyl reflections of gravity line

- don't change the global Lorentz signature (t, s)
- change only the identification of time

Weyl reflection $W_{\alpha_{11}}$

- change the global signature $(1, 10)$
- $(1, 10)$, $(2, 9)$, $(5, 6)$, $(6, 5)$, $(9, 2)$

Keurentjes '04

Link with Physics :

- String interpretation of $W_{\alpha_{11}}$ = double T-duality in the direction 9 and 10 + exchange of these directions

  Obers, Pioline '98
  Englert, Houart, Taormina, West '03

- E_8^{++}-invariant action contain in addition to M-Theory branes solutions, the exotic branes related to M' and M* theory

  Hull '98

$\Longrightarrow$ Signatures found for all $\mathcal{G}^{+++}$

de Buyl, Houart, Tabti '05
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Conclusion and perspectives

- Lot of properties of space-time theories neatly encoded in Kac-Moody algebras (branes, intersection rules, T-duality, ...)
- Study of $\mathcal{G}^{++}$ and $\mathcal{G}^{+++}$ constitute an interesting approach to understand gravitational theories coupled to matter which is conceptually different from the Einstein approach
- Fundamental question : are these symmetries only a consequence of the compactification process or are there effectively symmetries of the uncompactified theory ?
- Signifiance of the infinite tower of fields is important to give an answer to this question.

Englert, Houart, Kleinschmidt, Nicolai, Tabti 'work in progress'

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