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Noncommutative Symmetries and Gravity

Paolo Aschieri

Centro Fermi and U. Alessandria

- P. A., Blohmann, Dimitrijevic, Meyer Schupp, Wess hep-th/0504183 Class. Quant. Grav. (2005)
- P. A., Dimitrijevic, Meyer, Wess hep-th/0510059 Class. Quant. Grav. (2006)
- P. A. hep-th/0608172 (Corfu proceedings)

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- $$x^\mu * x^\nu - x^\nu * x^\mu = i\,\theta^{\mu\nu}$$

fundamental dimensionful constants, like c or $\hbar$.

- symmetries of these spaces

  e.g. coordinate transformations (diffeomorphisms)

  - Poincaré transformations

- Implement the principle of general covariance on noncommutative spaces and thus arrive at Einstein's equations for gravity on noncommutative manifolds.

Motivations ; noncommutativity from string theory and :

- Classical mechanics $\longrightarrow$ Quantum mechanics
  Observables become noncommutative

  Classical Gravity $\longrightarrow$ Quantum Gravity
  Space-time coordinates become noncommutative

- Gedanken experiment for testing space-time structure.

Class. Mech.

$\lambda = \frac{\hbar}{mc}$

Quant. Mech.

$L_p = 10^{-33}$ cm

Below L_p it is natural to conceive a more general, noncommutative, spacetime structure, where uncertainty

Noncommutativity of spacetime M is introduced via a $*$-product.

Usual product on $\mathcal{F}un(M)$

$$h \otimes g \xrightarrow{\cdot} h \cdot g$$

$*$-product on $\mathcal{F}un(M)$

$$h \otimes g \xrightarrow{*} h * g$$

$$(h * g)(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}} h(x)\, g(y) \Big|_{x=y}$$

$$\mathcal{F} = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}}$$

Drinfeld Twist $\mathcal{F}$

1983 1990

$$h \otimes g \xrightarrow{\mathcal{F}} \mathcal{F}(h \otimes g) \xrightarrow{\cdot} h * g$$

$$* = \cdot \circ \mathcal{F}$$

$$\tilde{\mathcal{F}} = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}$$

$$\tilde{\mathcal{F}} = 1\otimes 1 + \frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu + \frac{1}{2}\left(\frac{i}{2}\right)^2 \theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}\partial_{\mu_1}\partial_{\mu_2}\otimes\partial_{\nu_1}\partial_{\nu_2} + \dots$$

$$= f^\alpha \otimes f_\alpha$$

$$h * g = hg + \frac{i}{2}\theta^{\mu\nu}\partial_\mu h\, \partial_\nu g + \dots$$

- $$h * g = f^\alpha(h)\, f_\alpha(g)$$

More general twist $\mathcal{F}$, on arbitrary manifold M,

$$\mathcal{F} = e^{\frac{i}{2}\theta^{\mu\nu} X_\mu \otimes X_\nu}$$

where X_μ and X_ν are commuting $[X_\mu, X_\nu] = 0$.

The $*$- product is noncommutative

$$h * g \neq g * h$$

however it is "twisted" commutative

$$h * g = R^{\alpha}(g) * R_{\alpha}(h)$$

where

$$R = e^{-i\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} = R^{\alpha} \otimes R_{\alpha}$$

noncommutativity is controlled by R.

R gives a representation of the permutation group on $Fun_*(M)$.

$*$-Vectorfields.

$h \otimes v \to hv$ is deformed into

$$h * v = \bar{f}^{\alpha}(h)\, \bar{f}_{\alpha}(v)$$

$\partial_\nu(v) = \underbrace{[\partial_\nu, v]}_{\text{this is again a vector.}}$.

Coordinate expression:

$$v = v^\mu \partial_\mu$$

$$h * v = (h * v^\mu)\ \partial_\mu$$

$*$-Tensorfields.

$$\tau \otimes_* \tau' = \bar{f}^{\alpha}(\tau) \otimes \bar{f}_{\alpha}(\tau')$$

Coordinate expression:

$$\tau = \tau^{\mu_1 \dots \mu_n}\, \partial_{\mu_1} \otimes \dots \partial_{\mu_n}$$

$$\tau' = \tau'^{\nu_1 \dots \nu_q}\, \partial_{\nu_1} \otimes \dots \partial_{\nu_q}$$

$$\tau \otimes_* \tau' = \left(\tau^{\mu_1 \dots \mu_n} * \tau'^{\nu_1 \dots \nu_q}\right) \partial_{\mu_1} \otimes \dots \partial_{\mu_n} \otimes \partial_{\nu_1} \otimes \dots \partial_{\nu_q}$$

$\star$ - Action of vectorfields on functions

$$\delta_v(f) = v(f)$$ usual Lie derivative.

$$\delta : \text{Vectorfields} \times \text{Fun}(M) \longrightarrow \text{Fun}(M)$$
$$(v, f) \longmapsto v(f)$$

$$\delta^\star_v(f) := \bar{f}^\alpha(v)\left(\bar{f}_\alpha(f)\right)$$ $\star$-Lie derivative.

If $v = v^\mu \partial_\mu$ then

$$\delta^\star_v(f) = \underbrace{\bar{f}^\alpha(v^\mu \partial_\mu)}_{\bar{f}^\alpha(v^\mu)\,\partial_\mu}\left(\bar{f}_\alpha(f)\right) = \bar{f}^\alpha(v^\mu)\,\bar{f}_\alpha\left(\partial_\mu(f)\right) = v^\mu \star \partial_\mu f$$

The definition of $\delta^\star_v$ is good because

$$\delta^\star_{h\star v}(f) = h \star \delta^\star_v(f)$$ $\text{Fun}_\star(M)$ - linearity

On the $*$-product of two functions we have:

$$\delta^*_v(h*g) \neq \delta^*_v(h)*g + h*\delta^*_v(g)$$

but

$$\delta^*_v(h*g) = \delta^*_v(h)*g + R^\alpha(h)*\delta^*_{R_\alpha(v)}(g)$$ $*$-Leibniz rule.

$$v \longrightarrow v\otimes id + R^\alpha \otimes R_\alpha(v)$$ This defines a coproduct on $*$-algebra of vectorfields

$$u*v := f^\alpha(u)\, f_\alpha(v)$$

We have the $*$-Lie algebra of vectorfields

$$[u,v]_* = u*v - R^\alpha(v)*R_\alpha(u)$$

Thm: We have constructed the symmetry algebra (Hopf algebra) of $*$-deformed vectorfields and its $*$-Lie algebra, ($[u,v]_*$ is the $*$-Lie derivative of u on v).

Given a deformation $Fun_*(M)$ of the algebra of functions $Fun(M)$, for example, for $M = \mathbb{R}^4$

$$x^\mu * x^\nu - x^\nu * x^\mu = i\,\theta^{\mu\nu}$$

we can:

1) look for undeformed infinitesimal transformations that leave invariant $\theta^{\mu\nu}$ (where in general $\theta^{\mu\nu}$ transform as a tensor).

These undeformed infinitesimal transformations are the undeformed derivations of the algebra $Fun_*(M)$

2) deform the notion of infinitesimal transformation (derivation), so that

- there is a consistent underlying math. structure
- to any undeformed infinitesimal transformation there corresponds one and only one deformed infinitesimal transformation: $\delta_\varepsilon \longrightarrow \delta^*_\varepsilon$

- Usual Poincaré invariance is lost because

$$x^\mu * x^\nu - x^\nu * x^\mu = i\,\theta^{\mu\nu}$$

is not compatible with usual Lorentz transformat.

$$M_{\rho\sigma}\left(x^\mu * x^\nu - x^\nu * x^\mu\right) = M_{\rho\sigma}(x^\mu) * x^\nu + x^\mu * M_{\rho\sigma}(x^\nu) - \mu \leftrightarrow \nu$$

$$\neq 0$$

$$M_{\rho\sigma}\left(i\,\theta^{\mu\nu}\right) = 0$$

- However we have noncommutative Poincaré invariance! We do not loose space-time symmetry principles, we just deform them (this is not so for lattice space-time).

[Wess]

[Chaichian
Kulish
Nishijima
Tureanu
(2004)]

Notation

$$[A,B]_* = A * B - R^\alpha(B) * R_\alpha(A)$$

$$[A \overset{*}{,} B] = A * B - B * A$$

*-Poincaré Lie algebra

- $$[P_\mu, P_\nu]_* = [P_\mu \overset{*}{,} P_\nu] = 0$$
- $$[P_\rho, M_{\mu\nu}]_* = [P_\rho \overset{*}{,} M_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu)$$
- $$[M_{\mu\nu}, M_{\rho\sigma}]_* = -i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho})$$
  $$\neq [M_{\mu\nu} \overset{*}{,} M_{\rho\sigma}]$$
- $$P_\mu \longrightarrow P_\mu \otimes id + id \otimes P_\mu$$
- $$M_{\mu\nu} \longrightarrow M_{\mu\nu} \otimes id + id \otimes M_{\mu\nu} + i\theta^{\alpha\beta} P_\alpha \otimes [P_\beta, M_{\mu\nu}]$$

P.A. hep-th/0608133

- Covariant derivative

$$\nabla^*_\mu(v^\nu) = \partial_\mu(v^\nu) + v^\sigma \star \Gamma_{\mu\sigma}{}^\nu$$

$$\nabla^*_\mu(\omega_\nu) = \partial_\mu(\omega_\nu) - \Gamma_{\mu\nu}{}^\sigma \star \omega_\sigma$$

If $u = u^\rho \partial_\rho$

$$\nabla^*_u(v^\nu) = u^\rho \star \nabla_\rho v^\nu$$

- Curvature

$$h \star R(u, v, z) = \nabla^*_u \nabla^*_v z - \nabla^*_{R^\alpha(v)} \nabla^*_{R_\alpha(u)} z - \nabla^*_{[u,v]_\star} z$$

$$R(\partial_\mu, \partial_\nu, \partial_\rho) = R_{\mu\nu\rho}{}^\sigma \partial_\sigma$$

$$R_{\mu\nu\rho}{}^\sigma = \partial_\mu \Gamma_{\nu\rho}{}^\sigma - \partial_\nu \Gamma_{\mu\rho}{}^\sigma + \Gamma_{\nu\rho}{}^\beta \star \Gamma_{\mu\beta}{}^\sigma - \Gamma_{\mu\rho}{}^\beta \star \Gamma_{\nu\beta}{}^\sigma$$

[P.M. Dimitrijevic, Meyer, [illegible], Wess]

Metric and noncommutative gravity equations

$g_{\mu\nu}$

$g^{*\mu\nu}$ is the $*$-inverse metric

$$g^{*\mu\nu} * g_{\nu\rho} = g_{\rho\nu} * g^{*\nu\mu} = \delta^{\mu}_{\rho}$$

$$\Gamma_{\mu\nu}{}^{\rho} = \frac{1}{2}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) * g^{*\sigma\rho}$$

$R_{\mu\nu} = R_{\rho\mu\nu}{}^{\rho}$ Ricci tensor

$R_{\mu\nu} = 0$ Einstein equations on noncommutative spacetime

Conclusions & applications

- Dynamical noncommutativity
- Study of solutions of noncommutative gravity

It maight be that essential singularities are removed.

In particular I find interesting to study noncomm. gravitational instantons

- New manifolds invariants
- Coupling of gravity to matter (spinors) first order formalism.

- This is also a way to study field theories on nc backgrounds that are space-time dependent. ~~background~~

- Energy momentum of matter : conservation laws in nc field theories