Unfrozen hyperscalars and Supersymmetry

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Hypermultiplets are rather simple:

{ hyperini λ
{ hyperscalars q

The geometry describing the coupling between the hyperscalars is Quaternionic Kähler

$n_v > 1$:

$$J^r J^s = -\delta^{rs} + \varepsilon^{rst} J^t$$

$$H(J^r X, J^s Y) = H(X, Y)$$

$$\nabla_X J^r = \underbrace{-2\, \varepsilon^{rst} A_X^s J^t}_{su(2)-\text{connection}}$$

Implies $\Longrightarrow$ H is Einstein

$\Longrightarrow$ $\mathrm{Hol} \subseteq Sp(1)\cdot Sp(n_v)$

if A is pure gauge, we recover hyperkähler.

$n_v = 1$: A 4-dimensional manifold, with metric H, is said to be Quaternionic Kähler if H is

- Einstein & Selfdual

N.B. Selfduality means that the Weyl tensor is selfdual!

Coupling to Sugra

- To the action only through a σ-model

$$S = \int \sqrt{g}\,[R + \dots + H_{XY}\,\partial q^X \partial q^Y]$$

- the susy variations of the hyperini

$$\delta \lambda^\alpha = U^{\alpha I}_X\, \cancel{\partial} q^X\, \epsilon_I$$

- the susy rule for the gravitini

$$\delta \psi_\mu = (\nabla_\mu + \partial_\mu q^X A_X)\,\epsilon + \dots$$

Effects: Following ideas by K.P. Tod, one assumes the existence of a solution to the KSE, and uses those to construct tensor-fields. These tensor fields obey 2 types of constraints

1) Non-differential. I.e. Fierz Identities

2) Differential Constraints: W.r.t. the case without hypers, this is where the misery starts: The $su(2)$ part in the gravitino variation turns up!

$\mathcal{N}=1$ $d=5$ Sugra:

Spinors ϵ_I $(I=1,2)$ are Symplectic M.

$$f = i\,\bar{\epsilon}_I \epsilon^I$$

$$V^a = i\,\bar{\epsilon}_I \gamma^a \epsilon^I$$

$$\Phi^r{}_{ab} = \sigma^r{}_I{}^J\, \bar{\epsilon}_J \gamma_{ab} \epsilon^I$$

And the Fierz identities imply amongst other things:

$$V_a V^a = f^2 \qquad V^a \Phi^r{}_{ab} = V^a\, {}^\star\Phi^r{}_{abc} = 0$$

$$\Phi^r{}_a{}^c \Phi^s{}_{cb} = -\delta^{rs}(\eta_{ab} f^2 - V_a V_b) - \varepsilon^{rst} f\, \Phi^t{}_{ab}$$

As is usual $\quad \nabla_{(a} V_{b)} = 0 \Rightarrow V^a \partial_a \equiv \partial_t$

$$ds^2 = f^2 (dt+\omega)^2 - f^{-1} h_{mn}\, dx^m dx^n$$

Φ^r only has m,n indices and induces a quaternionic structure on h_{mn}

Two very important differential Constraints are

(I) $$\partial_m q^X = \sum_r \Phi^r{}_m{}^n \partial_n q^Y J_Y{}^X$$

(II) $$D_m \Phi^r{}_{np} = -2\,\varepsilon^{rst} A_m{}^s \Phi^t{}_{np}$$

(I) States that q is a quaternionic map. One can see that (I) implies that q is also harmonic, which is also the condition imposed by its equation of motion:

Before I forget: In the timelike case all the eq. of. motion are satisfied for a configuration that solves the KSEs.

(II) States that part of the spin connection of h_{mn} is related to the pull back of the su(2) connection on the hypervariety

Oh Good! The problem is interwoven

There are solutions, though.

Variations on a Solution by Jong, Kaya & Sergin

hep-th/0608034

Consider the metric

$$ds^2 = \frac{dq^x\, dq^x}{(1+\lambda q^2)^2} \qquad x = 1, \ldots 4 \qquad q^2 = \Sigma_x\, q^x q^x$$

$\lambda = +1 \qquad S^4 = SO(5)/SO(4)$

$\lambda = -1 \qquad H_4 = SO(1,4)/SO(4)$ Case treated by JKS

Both of these are quaternionic Kähler.

Needed later: $A^r = -\lambda \dfrac{q^x}{1+\lambda q^2} J^r_{x4}\, dq^4$

where J^r_{x4} is the quaternionic structure on flat space.

Take as the metric on the Base space another conformally flat one

$$h_{mn}\, dx^m dx^n = \Omega^2(x)\, dx^m dx^m$$

And take the quaternionic structure to be the same as on S^4/H_4.

then $\boxed{q^m(x) = \dfrac{x^m}{x^4}}$ $\qquad [x^4 = (x^m x^m)^2]$

Ω is fixed by the mathing of connections to be

$$\Omega = (1 + \lambda q^2)^{1/3} = \left(1 + \frac{\lambda}{x^6}\right)^{1/3}$$

$$\rightarrow \quad ds^2_h = \left(1 + \frac{\lambda}{r^6}\right)^{2/3} \left[dr^2 + r^2 dS^3 \right]$$

<u>Geometry</u>:

$\lambda = 1$: The possible singularity at $r = 0$ is illusory: all curvature invariants are regular everywhere and the limiting geometry is but $\mathbb{R}^4$.

$\lambda = -1$: There is a true unprotected curvature singularity at $r = 1$

So, what about solutions to Sugra... yes, I think I'd better say something...

$N=1$ $d=5$ Sugra – no vector multiplet
– hypers on S^4/H_u.

A supersymmetric Solution in this Setting is given by (more general ones are possible)

$$ds^2 = H^{-2}(dt+\omega)^2 + H\left(1+\frac{\lambda}{r^6}\right)^{2/3}(dr^2 + r^2 dS^3)$$

$$A = -\sqrt{3}\; K^{-1}\, dt$$

$$q^m = \frac{x^m}{x^4} = \frac{x^m}{r^4}$$

ω is restricted by $f d\omega = -*[f d\omega] \Rightarrow \omega = 0$

H has to be harmonic w.r.t. h_{mn}.

$$H = 1 + Q\,\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{4}{3};-\lambda r^{-6}\right)}{r^2}$$

${}_2F_1$ is a Gauß hypergeometric function..

Some properties:

$\Rightarrow$ ${}_2F_1(a,b;c;0)$ is a constant

$$\Rightarrow \lim_{r\to\infty} H = 1$$

$\Rightarrow$ $\dfrac{{}_2F_1(\frac{1}{3},\frac{2}{3};\frac{4}{3};-\lambda r^{-6})}{r^2}$ is monotonically decreasing.

What about $r = 0, 1$??

First $\lambda = -1$: then we are interested in

$$ {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{4}{3}; 1\right) = \frac{\Gamma(1/3)\,\Gamma(4/3)}{\Gamma(2/3)} $$

$$ \simeq 1.76664 $$

Nice, but remember the curvature singularity.

$\lambda = 1$: The calculation is a bit harder but the surprising result is

$$ \lim_{r \to 0} \frac{{}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{4}{3}; -r^{-6}\right)}{r^2} = \frac{\Gamma(1/3)\,\Gamma(4/3)}{\Gamma(2/3)} $$

So if $\lambda = 1$ and $\varphi > 0$, which is needed for positive mass, then our solution is Completely Regular.

A Quick Overview

$$
\begin{tikzcd}
 & \text{Timelike} & \text{Null} \\
d=6 & & e^+e^- + B_4 \arrow[dl] \arrow[d] \\
d=5 & T^2 + B_4 \arrow[d] \arrow[r, dashed, leftrightarrow] & e^+e^- + B_3 \arrow[d] \arrow[dl] \\
d=4 & T^2 + B_3 \arrow[r, leftrightarrow, "c\text{-map}"'] & e^+e^- + B_2
\end{tikzcd}
$$