The Stability of D-term Strings

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Based on:

- P. S., A. Collinucci & A. Van Proeyen hep-th/0610...

Outline

Introduction

A remarkable resurgence in interest in cosmic strings due to tentative evidence:

- double galaxy lensing candidate CSL-1 observation. Sazhin et al '03 X $\rightarrow$ Hubble
- double quasar Q0957+561A,B anomalous fluctuations. Schild et al '04

This was matched by considerable theoretical advances:

- F- and D-strings, (p,q)-string networks in string theory. Copeland, Myers & Polchinski, '03
- Solitonic string solution of Maxwell-Higgs theory embedded in $\mathcal{N} = 1$ supergravity - D-term string. Dvali, Kallosh & Van Proeyen, '03

Stability of 'cosmic superstrings' is crucial if we hope to observe them. Perturbative stability is implied by Bogomol'nyi bounds, but other decay channels exist e.g. to monopoles (Schwinger).

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N =1 Supergravity with F.I. terms

The bosonic sector is described by Einstein-Maxwell-Higgs Lagrangian:

$$e^{-1}\mathcal{L} = \frac{1}{2}R - \hat{\partial}_\mu\phi\,\hat{\partial}^\mu\phi^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V^D$$

ϕ is the $U(1)$-charged Higgs field with D-term potential,

$$V^D = \frac{1}{2}D^2 \qquad D = g\xi - g\phi^*\phi$$

W_μ is an abelian gauge field,

$$F_{\mu\nu} \equiv \partial_\mu W_\nu - \partial_\nu W_\mu\,, \qquad \hat{\partial}_\mu\phi \equiv (\partial_\mu - \mathrm{i}gW_\mu)\phi$$

i.e. Kähler potential $K = \phi^*\phi$, vanishing superpotential $W = 0$ and ξ is a constant Fayet-Iliopoulos term.

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The supersymmetry transformations for the gravitino ψ_μ, dilatino χ and gaugino λ are

$$\delta\psi_{\mu L} = \hat{\nabla}_\mu \epsilon_L = \left(\nabla_\mu + \tfrac{1}{2}\mathrm{i}A_\mu^B\right)\epsilon_L$$

$$\delta\chi_L = \tfrac{1}{2}(\not{\partial} - \mathrm{i}g\ \not{W})\phi\epsilon_R$$

$$\delta\lambda = \tfrac{1}{4}\gamma^{\mu\nu}F_{\mu\nu}\epsilon + \tfrac{1}{2}\mathrm{i}\gamma_5 D\epsilon$$

A_μ^B is a composite $U(1)$, defined by

$$A_\mu^B = \frac{1}{2}\mathrm{i}\left[\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi\right] + W_\mu D$$

Note we used the gravitino variation to define a supercovariant derivative $\hat{\nabla}_\mu\epsilon_L$ - this will play an important role later.

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The D-term string solution

Ansatz for cylindrically symmetric metric:

$$ds^2 = -dt^2 + dz^2 + dr^2 + C^2(r)d\theta^2$$

with the plane of the string parametrised by r and θ.

$$\phi(r,\theta) = f(r)\,e^{in\theta}$$

$f(r) \in \mathbb{R}$ and approaches the vacuum value $f^2 = \xi$ $(D = 0)$ outside the string core.

$$F = \tfrac{1}{2}F_{\mu\nu}\,dx^\mu\,dx^\nu = \frac{n\alpha'(r)}{g}dr\,d\theta$$

This ansatz solves $\delta\psi_{\mu L} = 0$ (and $\delta\chi_L = 0 = \delta\lambda$) if we impose

$$\gamma^{12}\epsilon = \mp i\gamma_5\epsilon$$

and demand that the following BPS condition on holds

$$1 - C'(r) = \pm A_\theta^B$$

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Some comments on the D-term solution

- This solution is preserves $1/2$ supersymmetry.

- However, explicit solutions for profile functions $\alpha(r), f(r)$ and $C(r)$ only known in limiting cases $r \to 0\ ,\ \infty$. e.g. for large r one finds $\alpha = 1$, and the metric takes the expected conical form:

$$C(r) = r\,(1 \mp n\xi)$$

  with the deficit angle $\delta = n\xi$ determined by the F.I. term ξ.

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Energy of string solutions - boundary terms Dvali et al ‘03, Achucarro et al ‘05

The canonical energy definition for field theory solitons employs an energy functional

$$\mu_{\rm string} = \int {\rm d}r{\rm d}\theta\, C(r)\left\{ |(\hat{\partial}_r\phi \pm {\rm i}C^{-1}\,\hat{\partial}_\theta)\phi|^2 + \frac{1}{2}\left[F_{12} \mp D\right]^2\right\}$$
$$+ \int {\rm d}r{\rm d}\theta \left[\partial_r \left(C' \pm A_\theta\right)^B \mp \partial_\theta A_r^B\right] - \int {\rm d}\theta\, C'\Big|_{r=\infty} + \int {\rm d}\theta C'\Big|_{r=0},$$

having inserted the solution and rearranged a la Bogomol’nyi. The BPS conditions mean that only the boundary terms remain.

$$\mu_{\rm string} = 2\pi\left(C'\big|_{r=0} - C'\big|_{r=\infty}\right) = \pm 2\pi n\xi : \text{ deficit angle}$$

This Bogomol’nyi approach is sufficient to define energy, but fails when one considers stability as it assumes cylindrical symmetry throughout e.g. non-axisymmetric perturbations would not be included.

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Refining the definition Hawking & Horowitz '95

The first step is to give a more rigourous definition of energy, using the Hamiltonian approach. Decompose $M_4 = \mathbb{R} \times \Sigma_t$, and define the physical Hamiltonian of the theory:

$$\mathcal{H}_{phys} \equiv \mathcal{H} - \mathcal{H}_0 = \int_{\Sigma_t} [NH + N_i H^i] - \int_{S_t^\infty} N({}^2K - {}^2K_0)$$

N, N_i - lapse and shift functions

H, H^i - canonical constraints

$\mathcal{H}_0$ - background Hamiltonian

The Gibbons- Hawking boundary terms have been rewritten as the extrinsic curvature ${}^2K_{(0)}$ of an asymptotic 2-surface S_t^∞ within the family of surfaces Σ_t.

Agrees with the known result:

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Improving the Bogomol'nyi argument - spinorial stability analysis

- D-term strings are 1/2 supersymmetric, $\therefore$ one would expect a positive energy theorem to apply $\rightarrow \delta\psi_u \sim$ Witten-Nester condition.

- However, $\delta\psi_u \Rightarrow E = \delta = \pm 2\pi n\xi$ - is this a contradiction?

- No - recall Schwarzschild: $M < 0$ is a solution, but it is ruled out of original positive energy theorem as $\nexists$ a Cauchy surface. For $M > 0$ one works outside the horizon where one can find a Cauchy surface.

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Getting rid of $\delta < 0$ Comtet and Gibbons '88

We can apply a result of Gibbons and Comtet:

- For cylindrically symmetric spacetimes $\exists$ a useful rewriting of the spatial component of the metric in which one can apply the Gauss-Bonnet theorem to simplify the initial value constraint:

$$\delta = \int_{\Sigma_2} T^0_0 + (\cdots)^2 \quad \Rightarrow \quad \delta < 0 = T^0_0 < 0$$

i.e. $\delta < 0$ violates the dominant energy condition, a key ingredient in the original positive energy theorem.

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Generalised Witten-Nester Energy

Using $\delta\psi_\mu$ as a guide, we define the generalised Witten-Nester tensor

$$\hat{E}^{\mu\nu} = \overline{\eta}\gamma^{\mu\nu\rho}\hat{\nabla}_\rho\eta + \text{h.c.}$$

where η is a commuting spinor that is asymptotically Killing, i.e.

$$\lim_{r\to\infty} \hat{\nabla}_\rho\eta = 0$$

Define a covariant surface integral[1], and its volume form

$$E_{\text{W.N.}} = \int d\Sigma_{\mu\nu}\hat{E}^{\mu\nu} = \int dV_\nu \nabla_\mu \hat{E}^{\mu\nu}$$

What is $E_{\text{W.N.}}$ is more familiar terms?

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Linearising the surface integral, and assuming sufficient fall-off conditions on η, one find that $E_{\text{W.N.}}$ gives the set of Killing charges [Witten, Nester '81]:

$$E_{\text{W.N.}} = P_\mu v^\mu = \int_{\Sigma_2} \Delta\omega_\mu v^\mu - Q_{\text{R}}$$

where $\Delta\omega_\mu$ is the spin connection perturbation ($\sim \partial^\mu h_{\mu\nu}$) and Q_{R} is the R-charge of the D-term string i.e. the holonomy of A^B_μ.

Choosing a timelike Killing vector v^μ, we find

$$E_{\text{W.N.}} \sim \int_{S_t^\infty} \left(N({}^2K - {}^2K_0) \right) - Q_{\text{R}}$$

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Positivity of $E_{\text{W.N.}}$ = Stability

Proof of positivity follows in the usual manner from the volume integral:

$$\begin{aligned} E_{\text{W.N.}} &= \int dV_\nu \nabla_\mu \hat{E}^{\mu\nu} \\ &= \int dV_\nu \hat{\nabla}_\mu \overline{\eta} \gamma^{\mu\nu\rho} \hat{\nabla}_\rho \eta + \overline{\delta\lambda} \gamma^\nu \delta\lambda + \overline{\delta\chi} \gamma^\nu \delta\chi \end{aligned}$$

where $\delta\lambda$ and $\delta\chi$ are defined as in the supersymmetry variations, but with commuting spinors η.

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The quantity defined by $E_{\text{W.N.}}$ is then manifestly positive:

$$E_{\text{W.N.}} = \int_{\Sigma_2} \Delta\omega_\mu v^\mu - Q_{\text{R}} = \int dV\, \delta\psi^2 + \delta\lambda^2 + \delta\chi^2$$

and the BPS bound is reproduced

$$E_{\text{W.N.}} > 0 \quad \Rightarrow \quad E \geq Q_{\text{R}}$$

$\Rightarrow$ D-term string is stable

We have proved perturbative stability non-linearly. In fact, within this sector the proof is semi-classically non-perturbative[2], however that's most likely not the whole story…

[2] [3D SUGRA - Becker et al, Edelstein et al 95]

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Conclusions

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- The $\delta\psi_\mu$ cancellation allows both existence of non-trivial string solutions and a non-linear stability proof.
- Interesting? - while other decay channels may exist, understanding perturbative stability remains important .
- Could offer an interesting insight complicated situations - embeddings in $\mathcal{N} = 2$ supergravity and 6D 3-brane models (Salam-Sezgin).

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- Could offer an interesting insight complicated situations - embeddings in $\mathcal{N} = 2$ supergravity and 6D 3-brane models (Salam-Sezgin).

Consider a useful rewriting of the spatial component of cylindrically symmetric metric Comtet and Gibbons '88:

$$ds_3^2 = e^{2\sigma}\left(dz^2 + \omega_i dx^i\right)^2 + d\Sigma_2$$

all metric functions are z-indep., and $d\Sigma_2$ is the transverse 2-surface metric.

Applying Gauss-Bonnet theorem to Σ_2 one can see that $\delta < 0$ is ruled out from various perspectives:

$$\int_{\Sigma_2} K = \delta \quad \therefore \quad \delta < 0 \Rightarrow K < 0$$

Σ_2 folds-up on itself - c.f. trapped surfaces in singularity theorems $\theta \equiv K$, for timelike normal vector.

Dominant energy condition violation

$$\delta = \int_{\Sigma_2} T_0^0 + (\cdots)^2 \quad \Rightarrow \quad \delta < 0 = T_0^0 < 0$$

i.e. violates key ingredient in the positive energy theorem.