Some Exact Results in Four-dimensional Non-perturbative String Theory

BASED ON :

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① MOTIVATION

String theory effective actions generically receive quantum corrections from both α' and g_s

worldsheet CFT   string loops

Non-perturbatively

worldsheet   spacetime

In string compactifications to $D = 4$ with $N > 2$, these quantum corrections contribute to higher derivative terms.

For $N = 2$ and $N = 1$ they contribute also to the Low Energy

Example $(N=2)$

$$\begin{array}{ccc}
\text{IIA}/CY & \qquad & \text{IIB}/\widetilde{CY} \\
\swarrow \quad \searrow & & \swarrow \quad \searrow \\
VM + HM & & VM + HM \\
(\alpha') \quad (g_s) & & (-) \quad (\alpha', g_s
\end{array}$$

Vector Multiplet modul

space : Special Kähler

geometry $F(X) =$

The hypermultiplet moduli space has a complicated geometrical structure

$$S = \int d^4x \sqrt{g} \left[R - \frac{1}{2} g_{ij} \partial_\mu \phi^i \partial^\mu \phi^j + \cdots \right]$$

Target space is Quaternion-Kähler

(Bagger & Witten)

In non-perturbative string theory

$$\boxed{g_{ij}(\phi) = g_{ij}^{\text{class.}}(\phi) + g_{ij}^{\text{pert.}}(\phi) + g_{ij}^{\text{inst.}}(\phi)}$$

$\boxed{\text{Q}}$ Can we compute the instanton contributions

IIA : Even (Euclidean) branes wrapping odd cycles

IIB : Odd branes wrapping even cycles

e.g. D1 over γ_2.

(Becker, Becker, Strominger)

In this talk, I will determine

- WORLDSHEET INST.
- D(−1) INSTANTONS
- D1 "

(II) IDEA

Use the $SL(2,\mathbb{Z})$ symmetry of type IIB

$$S: \quad F1 \longleftrightarrow D1$$

worldsheet instantons $\longleftrightarrow$ D1 instanton (Witten)

$\uparrow$

Known from MIRROR symmetry (Candelas &

D(-1) instantons :

Use work by Green and Gutperle , ... about R^4 terms in $D = 10$

D-1 cannot probe the CY_3 , so we can compactify to $D = 4$.

$\boxed{Q}$ How to implement this idea in $D = 4$?

III SOME GEOMETRY

Dimension of HM

moduli space $= 4(h_{1,1}+1$

Quaternion-Kähler (QK

metric $(i = 1, \ldots, 4h_{1,1}+4)$

$$ds^2 = g_{ij}(\phi)\, d\phi^i\, d\phi^j$$

Inherits $SL(2,\mathbb{R})$

symmetry (Isometry)

(Böhm, Günther,

Using the conformal calculus in N=2 SUGRA one can show there is a 1-to-1 correspondence between QK and conformal Hyperkähler geometry

HKC $\quad g_{AB}(\phi, \phi^\circ) \quad 4(n+1)$

$\Updownarrow$ compensator for $D \times SU(2)_R$ $\Updownarrow$

Moreover, g_{AB} is determined by a SINGLE function

$$\boxed{g_{AB} = D_A \partial_B \chi}$$

- de Wit, Kleijn, V.
- de Wit, Roček, V.
- Swann ; Galicki
- Bergshoeff, Cucu, T. de Wit
  T. Gheerardyn, S.V., Van

Any isometry of the QK is lifted to an invariance of $\chi(\phi, \phi^{\circ})$

$$\boxed{\delta_G \, \chi = 0}$$

For $G = SL(2, \mathbb{Z})$, this implies that χ is

MODULAR INVARIANT

$\Rightarrow$ Key to the solution of the problem!

IV RESULTS

At tree-level in α' and g_s, we determined

$$\boxed{\chi_T^{class} = 4\, r^0 \, (\mathrm{Im}\,\tau)^2 \, V(z-\bar{z})}$$

r^0 : compensator

τ : dilaton-axion complex

$$V(z-\bar{z}) = \frac{i}{48}\, d_{abc}\, (z-\bar{z})^a (z-\bar{z})^b (z-\bar{z})^c$$

$\Rightarrow$ $SL(2,\mathbb{R})$ invariant

Perturbative corrections in α' and g_s :

$$\chi^{pert} = \underbrace{4\pi^2 (\mathrm{Im}\,\tau)^{1/2}}_{\text{MODULAR INV.}} \chi_{EULER} \times$$

$$\times \underbrace{\left(\frac{1}{4} \zeta(3) (\mathrm{Im}\,\tau)^{3/2} + \frac{1}{2} \zeta(2) (\mathrm{Im}\,\tau)^{-1/2} \right)}_{\text{Not modular inv !}}$$

$$S : \quad \mathrm{Im}\,\tau \to \frac{\mathrm{Im}\,\tau}{|\tau|^2}$$

There is a UNIQUE way to complete this into a modular invariant consistent with susy :

$$\boxed{\chi = 2\pi\, \chi_{Euler}\ r^{0}\ (\mathrm{Im}\,\tau)^{1/2} \times E_{3/2}(\tau, \bar{\tau})}$$

The Eisenstein series can be expanded and includes a sum over all (D-1) instantons & anti-instantons

With $\tau = \tau_1 + i\,\tau_2$

$$\left.\begin{array}{l} 4\pi\, E_{3/2} = 2\,\zeta(3)\,\tau_2^{3/2} \\[2em] \qquad\qquad + \dfrac{2\pi^2}{3}\,(\tau_2)^{-1/2} \end{array}\right\} \begin{array}{l}\text{pert.}\\ \text{loops}\end{array}$$

$$+\, 4\pi^{3/2} \sum_{m,n\geq 1} \left(\frac{m}{n^3}\right)^{1/2} \Big(\underset{D-1}{e^{2\pi i m \tau}} + \underset{\overline{(D-1)}}{c.c.} \Big)$$

$$\times \left[1 + \sum_{k=1}^{\infty} \frac{\Gamma(k-1/2)}{\Gamma(k+1/2)}\, \frac{1}{(4\pi m n \tau_2)^k} \right]$$

$$\uparrow$$

Fluctuations around the instantons

<u>The</u> <u>real</u> <u>stuff</u> :

applying the c-map on the worldsheet instanton corrections, we get

$$\boxed{\chi = -2\, r^{0} (\tau_2)^2 \sum_{d_a} n_{d_a} \left[Li_3(e^{iw}) + w_2\, Li_2(e^{iw}) + c.c. \right]}$$

$$w = w_1 + i\, w_2 = 2\pi\, d_a\, z^a$$

$$Li(x) = \sum^{\infty} \frac{x^n}{}$$

This function is not $SL(2,\mathbb{Z})$ invariant because D1-instantons are not included.

One can apply the METHOD OF IMAGES to make it modular invariant

$$\tilde{f}(\tau,\bar{\tau}) = \sum f(g\tau, g\bar{\tau})$$

The result of this method gives

$$\boxed{\begin{aligned} \chi^{inst} &= -2^{0} \, (\tau_2)^{1/2} \sum_{d_a} n_{d_a} \times \\ &\times \sum_{(m,n)} \frac{(\tau_2)^{3/2}}{|m\tau + n|^3} \left(1 + |m\tau + n| \, W_2 \ldots\right. \\ &\qquad \times \, e^{-S_{m,n}} \end{aligned}}$$

SUM OVER "BOUND STATES" OF MIXED

INSTANT... ACTION

The instanton action gives the right answer for D1-instantons when $n=0$

$$S = \frac{|m|}{g_s} + i\, d_a A^a$$

RR scalars

Modular invariance:

$T: \; S_{m,n} \longrightarrow S_{m,n+m}$

$S: \; S \longrightarrow S$