# Supersymmetric Black Holes in String Theory

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# Outline

1. Introduction: the laws of black mechanics, the string–black hole correspondence and BPS states.
2. BPS black holes in $\mathcal{N} = 2$ compactifications: special geometry, attractor mechanism, variational principle, and higher derivative corrections. With comments on nonsupersymmetric black holes.
3. OSV conjecture, large and small black holes.
4. Concluding remarks.

# The Laws of Black Hole Mechanics

(0) $\kappa_S =$ const.

(1) $\delta M = \frac{\kappa_S}{8\pi}\delta A + \omega\delta J + \phi\delta q$.

(2) $\delta A \geq 0$.

(3) $\kappa_S = 0$ cannot be reached in finite time by any physical process.

$ks =$ surface gravity, $M =$ mass, $A =$ horizon area, $\omega =$ horizon angular velocity, $J =$ angular momentum, $\phi =$ chemical (= electrostatic) potential, $q =$ electric charge.
`J.M. Bardeen, B. Carter and S.W. Hawking (1973)`

Suggests:

$$\kappa_S \sim T \ , \quad A \sim \mathrm{S}$$

where $T =$ temperature and $\mathrm{S} =$ Entropy.

Assumptions: (0), (1): stationary black hole, (2): 'predictable' space-time, Einstein's field equations satisfied, (dominant/weak) energy condition.

However, (0), (1) hold irrespective of the details of the field equations, if appropriate symmetry conditions are imposed!

# The Laws of Black Hole Mechanics (2)

Modified assumptions: generally covariant Lagrangian with black hole solution such that

(i) black hole is static or stationary, axisymmetric, $t-\phi$ reflection symmetric,

(ii) event horizon is a Killing horizon,

(iii) a Cauchy hypersurface exists.

Then

(0) $\kappa_S =$ const.

`R. Wald and Racz (1995)`

(1) $\delta M = \frac{\kappa_S}{2\pi}\delta \mathrm{S} + \omega \delta J + \phi \delta q$.

`R. Wald (1993), ...`

provided that $M, J, \mathrm{S}, q$ are defined as appropriate surface charges.

Entropy:

$$\mathrm{S} = \int_H Q[\xi]$$

$\xi =$ 'horizontal' Killing vector field, $Q =$ Noether two-form.

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provided that $M, J, \mathrm{S}, q$ are defined as appropriate surface charges.

Entropy:

$$\mathrm{S} = 2\pi \int_H \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \varepsilon^{\mu\nu} \varepsilon^{\rho\sigma} \sqrt{h} d^2\Omega$$

$\varepsilon^{\mu\nu} =$ normal bivector of horizon, $\sqrt{h} d^2\Omega =$ induced volume form.

# Black Hole Thermodynamics

Hawking radiation:

$$T_{\text{Hawking}} = \frac{\kappa_S}{2\pi} \quad (G_N = c = \hbar = 1)\,.$$

First law:

$$\delta M = \frac{\kappa_S}{8\pi}\delta A + \cdots$$

$$\Rightarrow \mathrm{S} = \frac{A}{4}\ .$$

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$\mathrm{S} = \mathrm{S}_{\rm macro} =$ thermodynamical or macroscopic entropy. Unterlying microscopic theory (=quantum gravity) should specify the microscopic states of the black hole.

Microscopic entropy:

$$\mathrm{S}_{\rm micro} = \log d(M, J, q)\;, \quad d = \#\text{microstates}\;.$$

Expect: $\mathrm{S}_{\rm macro} = \mathrm{S}_{\rm micro}$ .

Benchmark for theories of quantum gravity!

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More ambitiously: derive black hole thermodynamics starting from a microscopic partition function.

Example: OSV conjecture for supersymmetric black holes.

# The String – Black Hole Correspondence

Idea: black hole microstates $=$ string states at large mass or large coupling.

Perturbative string regime $\leftrightarrow$ Semiclassical gravity regime.

$\sqrt{\alpha'} \gg r_S$ $\sqrt{\alpha'} \ll r_S$.

L. Susskind (1993), G.T. Horowitz and J. Polchinski (1997).

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| --- | --- | --- |
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Compare 4d Schwarzschild black hole to open bosonic string (truncated).

Relation of gravitational and string scale: $G_N = g_S^2 \alpha'$.

String mass formula:

$$\alpha' M^2 \approx n \quad \text{(for large excitation number } n\text{)} \ .$$

String entropy:

$$\mathrm{S}_{\mathrm{String}} = \log d(n) \sim \sqrt{n} \ .$$

Schwarzschild radius of string state:

$$r_S \approx g_S^2 \sqrt{n} \sqrt{\alpha'} \ .$$

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- Perturbative string in flat space:

$$g_S^2\sqrt{n} \ll 1 \Rightarrow r_S \ll \sqrt{\alpha'}\ , \quad \mathrm{S}_{BH} \ll \mathrm{S}_{\mathrm{String}}$$

- Semiclassical black hole:

$$g_S^2\sqrt{n} \gg 1 \Rightarrow r_S \gg \sqrt{\alpha'}\ , \quad \mathrm{S}_{BH} \gg \mathrm{S}_{\mathrm{String}}\ .$$

- Transition(?):

$$g_S^2\sqrt{n} \approx 1 \Rightarrow r_S \approx \sqrt{\alpha'}\ , \quad \mathrm{S}_{BH} \approx \mathrm{S}_{\mathrm{String}}\ .$$

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| | | |
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Need to interpolate between two accessible regimes. Intermediate regime not under control.
Matching of entropies up to $\mathcal{O}(1)$:

$$\mathrm{S}_{\mathrm{BH}} \sim \mathrm{S}_{\mathrm{String}} \; .$$

Consider supersymmetric (BPS) states. `A. Strominger and C. Vafa (1996), ...`
Interpolation more plausible.
Can compute $\mathrm{S}_{\mathrm{macro}} \equiv \mathrm{S}_{\mathrm{BH}}$ and $\mathrm{S}_{\mathrm{micro}} \equiv \mathrm{S}_{\mathrm{String}}$ to high precision in their respective regimes.
Find 'exact' matching

$$\mathrm{S}_{\mathrm{BH}} \approx \mathrm{S}_{\mathrm{String}} \; ,$$

including subleading corrections (in large mass $=$ semiclassical expansion).

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Besides fundamental strings, also solitonic $p$-branes are associated with microscopic degrees of freedom.

E.g. the pioneering work of A. Strominger and C. Vafa involved D-branes, rather than fundamental strings.

We will discuss examples involving fundamental (and also solitonic) strings later.

# The String – Black Hole Correspondence

Idea: black hole microstates $=$ string states at large mass or large coupling.

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| --- | --- | --- |
| $\sqrt{\alpha'} \gg r_S$ | | $\sqrt{\alpha'} \ll r_S$. |

Beyond matching numbers: OSV conjecture

$$Z_{\mathrm{BH}} \approx |Z_{\mathrm{top}}|^2 \ .$$

$Z_{BH} =$ black hole partition function, $Z_{\mathrm{top}} =$ partition function of the topological string.

`H. Ooguri, A. Strominger and C. Vafa (2004)`

# BPS states

Supersymmetry algebra (4d, Weyl spinors):

$$\{Q_\alpha, Q^+_\beta\} = 2\sigma^\mu_{\alpha\beta}$$

# BPS states

$\mathcal{N}$-extended supersymmetry algebra (4d, Weyl spinors):

$$\begin{aligned} \{Q^A_\alpha, Q^{+B}_\beta\} &= 2\sigma^\mu_{\alpha\beta}\delta^{AB} \\ \{Q^A_\alpha, Q^B_\beta\} &= \epsilon_{\alpha\beta} Z^{AB} \end{aligned}$$

$A, B, \ldots = 1, \ldots, N$.
Central charges $=$ skew eigenvalues of $Z^{AB}$:

$$M^2 \geq |Z_1|^2 \geq |Z_2|^2 \geq \cdots \geq 0 \ .$$

Saturation of inequalities $\Rightarrow$ shortened (BPS) multiplets.

# BPS states

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$A, B, \ldots = 1, \ldots, N$.

Examples:

- $\mathcal{N} = 2$:
  1. $M > |Z|$: generic massive multiplet.
  2. $M = |Z|$: short or $\frac{1}{2}$-BPS multiplet.
- $\mathcal{N} = 4$:
  1. $M > |Z_1| > |Z_2|$: generic massive multiplet.
  2. $M = |Z_1| > |Z_2|$: $\frac{1}{4}$-BPS multiplet.
  3. $M = |Z_1| = |Z_2|$: $\frac{1}{2}$-BPS multiplet.

Supersymmetric vacua $=$ '$\frac{1}{1}$-BPS' (fully supersymmetric).

# BPS solitons

BPS states can be realized as finite energy solutions $\Phi_0$ of the field equations (asymptotic to vacuum).

Killing spinors $\varepsilon \leftrightarrow$ residual (rigid) supersymmetry of $\Phi_0$:

$$\delta_\varepsilon \; \Phi|_{\Phi_0} = 0 \; .$$

Example: the extreme Reissner-Nordstrom black hole regarded as a solution of $\mathcal{N} = 2$ Supergravity = Einstein-Maxwell + 2 Gravitini.

Has 4 Killing spinors and interpolates between two supersymmetric vacua (8 Killing spinors): Minkowski space at infinity and $AdS^2 \times S^2$ (with covariantly constant gauge fields) at horizon.

`G. Gibbons (1982)`

# Embedding into string theory

String compactifications give supergravity plus matter.

We consider:

- Het$/(K3 \times T^2)$ and type-II/$CY_3$
  $\longrightarrow \mathcal{N} = 2$ Supergravity + $n_V$ vector multiplets ( + $n_H$ hypermultiplets + $n_T$ tensor multiplets ).
- Het$/T^6$ and type-II/$(K3 \times T^2)$
  $\longrightarrow \mathcal{N} = 4$ Supergravity + $n_V$ vector multiplets.

Main tool: specical geometry of $\mathcal{N} = 2$ vector multiplets.
B. de Wit and A. Van Proeyen (1984).

All vector multiplet couplings are encoded in a holomorphic prepotential.
Field equations are invariant under $Sp(2n_V + 2, \mathbb{R})$ rotations which generalize the electric-magnetic duality of Maxwell theory, and include stringy symmetries such as T-duality and S-duality.

# Special geometry (1)

Multiplets:

- Gravity multiplet: $\{e_\mu{}^A, \psi^i_\mu, \mathcal{A}_\mu\}$.
- Vector multiplet: $\{\mathcal{A}_\mu, \lambda^i, z\}^A$.

$i = 1, 2$, $A = 1, \ldots, n_V$.

Bosonic Lagrangian:

$$8\pi e^{-1}\mathcal{L}_{\rm bos} = -\frac{R}{2} - g_{A\overline{B}}(z,\overline{z})\partial_\mu z^A \partial^\mu \overline{z}^{\overline{B}} + \frac{i}{4}\overline{\mathcal{N}}_{IJ}(z,\overline{z})F^{-I}_{\mu\nu}F^{-I|\mu\nu} - \frac{i}{4}\mathcal{N}_{IJ}(z,\overline{z})F^{+I}_{\mu\nu}F^{+I|\mu\nu}$$

$I = 0, 1, \ldots, n_V$.
$F^{\pm I}_{\mu\nu}$ = (anti-)selfdual part of field strength.

To make electric-magnetic duality manifest, define dual field strength:

$$G^{\pm}_{I|\mu\nu} \simeq \frac{\delta\mathcal{L}}{\delta F^{\pm I|\mu\nu}} \, .$$

# Special Geometry (2)

Field equations (not: action) are invariant under $Sp(2n_V + 2, \mathbb{R})$ rotations.

Symplectic vectors:

- Gauge fields and charges: $\begin{pmatrix} F^{\pm I}_{\mu\nu} \\ G^{\pm}_{I|\mu\nu} \end{pmatrix}$ , $\begin{pmatrix} p^I \\ q_I \end{pmatrix} = \begin{pmatrix} \oint F^I \\ \oint G_I \end{pmatrix}$
- Scalars: $\begin{pmatrix} X^I \\ F_I \end{pmatrix}$

where 'scalars' $X^I$ are related to the physical scalars $z^A$ by $z^A = \frac{X^A}{X^0}$ and

$$F_I = \frac{\partial F}{\partial X^I} \ .$$

Prepotential $F(X)$ is holomorphic and homogenous of degree 2:

$$F(\lambda X^I) = \lambda^2 F(X) \ .$$

# Special Geometry (3)

| | | | | |
|---|---|---|---|---|
| Poincaré Supergravity | $\longleftrightarrow$ | Conformal Supergravity | | |
| $n_V$ vector multiplets | | $(n_V + 1)$ vector multiplets | | |
| $\mathcal{M}_{VM}$ | $\longleftrightarrow$ | $\mathcal{CM}_{VM}$ | $\stackrel{\Phi}{\longrightarrow}$ | $\mathbb{C}^{2n_V+2}$ |
| $z^A$ | | $X^I$ | | $\begin{pmatrix} X^I \\ F_I \end{pmatrix}$ |

$\mathcal{CM}_{VM}$ is a complex cone over $\mathcal{M}_{VM}$. (Prepotential needs to be homogenous).

(Holomorphic) Prepotential $F$ defines holomorphic Lagrangian immersion, $\Phi = dF$ into symplectic vector space $\mathbb{C}^{2n_V+2} = T^*\mathbb{C}^{n_V+1}$.

`Embedding construction explained in: D.V. Alekseevsky, V. Cortés and C. Devchand, math.dg/9910091.`

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Special geometry is naturally realised in type-II Calabi-Yau compactifications:

$\mathcal{M}_{VM}$ = complex structure moduli (IIB).
$\mathcal{CM}_{VM}$ = complex structure moduli + holomorphic volume form.
$\mathbb{C}^{2n_V+2} = H^3(X, \mathbb{C})$. Middle cohomology.

# Special Geometry (4)

Real parametrizations and Hesse potential.

Kähler manifold: complex and sympletic (compatible).
Special Kähler manifold: existence of a flat, torsion-free, symplectic connection, such that $\nabla_{[\mu} I^{\nu}{}_{\rho]} = 0$ . D.S. Freed (1997)

Associated Darboux coordinates:

$$x^I = \text{Re} X^I \ , \quad y_I = \text{Re} F_I \ .$$

Legendre transform $X^I = (\text{Re}(X^I), \text{Im}(X^I)) \rightarrow (x^I, y_I)$. All couplings encoded in the Hesse potential (real Kähler potential):

$$H(x^I, y_I) = 2 \left( \text{Im} F - (\text{Re} F_I)(\text{Im} X^I) \right) \quad \left( \text{note} \;\; \text{Re} F_I = \frac{\partial (\text{Im} F)}{\partial (\text{Im} X^I)} \right) .$$

V. Cortés (2001).

The real parametrization is very useful for BPS black holes and other BPS solitons.

# BPS Black Holes

Impose

- $\frac{1}{2}$-BPS solution, i.e. 4 linearly independent Killing spinors $\varepsilon_i$: $\delta_{\varepsilon_i}\Phi = 0$.
  $\Phi =$ all fields.
- Solution static and spherically symmetric.
  Metric:
  $$ds^2 = -e^{2g(r)}dt^2 + e^{2f(r)}(dr^2 + r^2 d\Omega^2)$$
  $\frac{1}{2}$-BPS $\Rightarrow g(r) = -f(r)$.
  Gauge fields (using orthonormal frame):
  $$F^I_{\underline{tr}} = F^I_E(r) \sim q_I \ , \quad F^I_{\underline{\theta\phi}} = F_M(r) \sim p_I$$
  Scalars:
  $$z^A = z^A(r) = \frac{X^A(r)}{X^0(r)} = \frac{Y^A(r)}{Y^0(r)}$$
  $X^I, Y^I$ are related by an ($r$-dependent) rescaling.

New features, compared to extreme Reissner-Nordstrom black hole: several gauge fields and charges, scalars.

# Attractor mechanism

1. $r \to \infty$: solution asymptotically flat.
   $z^A \to z^A(\infty) \in \mathcal{M}_{VM}$.
   Minkowski vacuum with arbitrary moduli.
2. $r \to 0$ (event horizon): $z^A \to z^A_*(p,q)$.
   Gauge fields and metric determined by $z^A_*(p,q)$:
   Solution is Bertotti-Robinson solution $AdS^2 \times S^2$ with radius fixed by the charges.
   Horizon solution = supersymmetric vacuum with fixed moduli.

Attractor mechanism! S. Ferrara, R. Kallosh and A. Strominger (1995).

Attractor equations:

$$\left( \begin{array}{c} Y^I - \overline{Y}^I \\ F_I - \overline{F}_I \end{array} \right)_* = i \left( \begin{array}{c} p^I \\ q_I \end{array} \right)$$

Entropy:

$$\mathrm{S}_{\mathrm{macro}} = \frac{A}{4} = \pi |Z|^2_* = \pi |p^I F_I(X) - q_I X^I|^2 = \pi \left( p^I F_I(Y) - q_I Y^I \right)_*$$

# Variational Principle

Complex version.

Entropy function:

$$\Sigma(Y,\overline{Y},p,q) := \mathcal{F}(Y,\overline{Y}) - q_I(Y^I + \overline{Y}^I) + p^I(F_I + \overline{F}_I) \; .$$

Free energy:

$$\mathcal{F}(Y,\overline{Y}) = -i(F_I\overline{Y}^I - Y^I\overline{F}_I) \; .$$

Critical points:

$$\frac{\partial\Sigma}{\partial Y^I} = 0 = \frac{\partial\Sigma}{\partial\overline{Y}^I} \iff \text{attractor equations}$$

Critical value $\propto$ entropy:

$$\pi\Sigma_* = \mathrm{S}_{\mathrm{macro}}(p,q) \; .$$

K. Behrndt, G.L. Cardoso, B. de Wit, R. Kallosh, D. Lüst and T.M. (1996)

# Variational Principle

Real version:

Entropy function:

$$\Sigma(x,y,q,p) = 2H(x,y) - 2q_I x^I + 2p^I y_I \ .$$

Free energy $\propto$ Hesse potential:

$$2H(x,y) = \mathcal{F}(Y,\overline{Y}) = -i(F_I \overline{Y}^I - Y^I \overline{F}_I) \ .$$

Critical points:

$$\frac{\partial H}{\partial x^I} = q_I \ , \quad \frac{\partial H}{\partial y_I} = -p^I \Leftrightarrow \quad \text{attractor equations .}$$

Entropy $=$ (full) Legendre transform of Hesse potential:

$$\mathrm{S}_{\mathrm{macro}}(p,q) = 2\pi \left( H - x^I \frac{\partial H}{\partial x^I} - y_I \frac{\partial H}{\partial y_I} \right)_*$$

G.L. Cardoso, B. de Wit, J. Käppeli and T.M., JHEP 03 (2006) 074, hep-th/0601108.

# $R^2$ Corrections

String-effective actions contain an infinite series of higher-derivative terms, computable (in principle) from string perturbation theory.

In $\mathcal{N} = 2$ supersgravity, there is an off-shell description for a particular class of such terms, (called '$R^2$-terms').

Conformal Supergravity: gravitational degrees of freedom reside in the Weyl multiplet.

Generalized prepotential with explicit dependence on the Weyl multiplet:

$$F(Y^I) \rightarrow F(Y^I, \Upsilon)$$

where $\Upsilon =$ lowest component of Weyl multiplet $\sim$ Graviphoton field strength.

$F(Y^I, \Upsilon)$ is holomorphic and (graded) homogenous of degree two:

$$F(\lambda Y^I, \lambda^2 \Upsilon) = \lambda^2 F(Y^I, \Upsilon) \ .$$

Weyl multiplet can be treated as a background field.

`B. de Wit, hep-th/9602060, 9603191.`

# $R^2$ Corrections (2)

Expand in $\Upsilon$:

$$F(Y^I, \Upsilon) = \sum_{g=0}^{\infty} F^{(g)}(Y^I)\Upsilon^g \ .$$

Higher derivative terms include:

$$\mathcal{L} \sim \sum_{g=1}^{\infty} \left( F^{(g)}(Y^I)(C^-_{\mu\nu\rho\sigma})^2 (T^-_{\alpha\beta})^{2g-2} + \text{c.c.} \right) + \cdots$$

$C_{\mu\nu\rho\sigma}$ = Weyl tensor, $T_{\mu\nu}$ = Graviphoton field strength.

In type-II CY compactifications, $F^{(g)}(Y^I)$ can be computed in the topologically twisted theory.
$F^{(g)} \propto$ genus $g$ contribution to topological free energy, i.e. $\exp(F^{(g)}) \propto$ partition function of topological string on genus g world sheet.

M. Beshadsky, S. Cecotti and H. Ooguri (1993).
I. Antoniadis, E. Gava, C. Narain and T.R.Taylor (1993).

# $R^2$ Corrections (3)

Attractor mechanism still works.
Solutions can be constructed, at least iteratively.
Off-shell formulation essential.

Attractor equations:

$$\begin{pmatrix} Y^I - \overline{Y}^I \\ F_I(Y,\Upsilon) - \overline{F}_I(\overline{Y},\overline{\Upsilon}) \end{pmatrix}_* = i \begin{pmatrix} p^I \\ q_I \end{pmatrix} , \quad \Upsilon_* = -64 \,.$$

Entropy:

$$\mathrm{S}_{\mathrm{macro}} = \mathrm{S}_{\mathrm{Wald}} = \pi \left( \left( p^I F_I(Y,\Upsilon) - q_I Y^I \right) - 256 \,\mathrm{Im} \left( \frac{\partial F}{\partial \Upsilon} \right) \right)_*$$

Note: modification of area and modification of area law.
Important for matching $\mathrm{S}_{\mathrm{macro}} = \mathrm{S}_{\mathrm{micro}}$.

G.L. Cardoso, B. de Wit and T.M. (1998), G.L. Cardoso, B. de Wit, J. Käppeli and T.M. (2000)

# Other higher derivative terms

The Weyl multiplet encodes a specific class of higher derivative terms:

$$R^2 F^{2g-2} + \text{susy transformed}\ .$$

Other higher derivative terms should contribute to the entropy as well.

Evidence for particular importance of $R^2$-terms: (i) relation to topological string, (ii) success in matching $\mathrm{S}_{\mathrm{macro}} = \mathrm{S}_{\mathrm{micro}}$. (However: no $R^2$ terms in toroidal type-II compactifications: need $R^4$-terms.)

Observation: one can substitute the Gauss-Bonnet term for the whole set of supersymmetric $R^2$-terms.

`K. Behrndt, G.L. Cardoso and S. Mahapatra, NPB 732 (2006) 200, hep-th/0506251, A. Sen, JHEP 03 (2006) 008, hep-th/0508042.`

Entropy appears to be robust!? Universality?

# Other higher derivative terms (2)

Progress on explicit construction of further higher derivative terms (using superconformal calculus).

B. de Wit and F. Saueressig, hep-th/0606148

Observe cancellations in supersymmetric backgrounds.

4d $\longrightarrow$ 5d lift gives black holes with $AdS^3 \times S^2$ horizon geometry. Matching of $\mathrm{S}_{\mathrm{macro}}$ and $\mathrm{S}_{\mathrm{micro}}$ can be related to anomalies, hence insensitive to details of field equations. $R^2$-terms seem to cover precisely the relevant contributions.

P. Kraus and F. Larsen, JHEP 09 (2005) 034, hep-th/0506176.

# Non-supersymmetric Black Holes

Attractor mechanism also works for extremal black holes (with $AdS^2 \times S^2$ horizon geometry), which are not supersymmetric. This is again independent of the details of the action, and holds for generally covariant higher derivative theories.
Elegant formalism, based on an entropy function and Wald's entropy formula:
`A. Sen, hep-th/0506177, hep-th/0508042, ...`

Relation between Sen's formalism and the variational principle reviewed here has been explored by G.L. Cardoso, B. de Wit and S. Mahapatra, see Gabriel's talk.

Why is $AdS^2 \times S^2$ more essential than supersymmetry? Plausibility argument: $AdS^2 \times S^2$ is a flux compactification on $S^2$. Flux generates scalar potential which fixes scalars. Applies to non-superysmmetric theories, but also supersymmetric theories can have non-supersymmetric vacua.

Examples for non-supersymmetric, extremal black holes in supersymmetric compactifications:
`K. Goldstein, N. Iizuka, R.P. Jena and S.P. Trivedi, Phys. Rev. D 72 (2005) 124021, hep-th/0507096, R. Kallosh, JHEP 12 (2005) 022, hep-th/0510024, ...`

# Non-supersymmetric Black Holes (2)

Discrepancies between 5d and 4d analysis reported by `B. Sahoo and A. Sen, hep-th/0603149` suggest that additional higher derivative terms become relevant in the non-supersymmetric case. Consistent with observations of `B. de Wit, F. Saueressig, hep-th/0606148.`

Complementary to explicit constructions of actions and solutions, one can use general arguments based on anomalies or on the AdS-CFT correspondence, which do not depend on details of the field equations.
In fact within AdS-CFT one can even consider non-extremal black holes in asymptotic AdS space-time (horizon geometry $\neq AdS^m \times S^n$, but space-time is asymptotic to $AdS^{m+n}$ at infinity).

Nice lectures on applications of AdS$_3$–CFT$_2$: `P. Kraus, hep-th/0609074`.

# OSV conjecture

Observation:

$$Y^I - \overline{Y}^I = ip^I \Rightarrow Y^I = \frac{1}{2}(\phi^I + ip^I)$$

where $\phi^I$ = electrostatic potential (equations of motion) = chemical potential for electric charge.

Observation: extremization of entropy function $\Sigma(\phi, p, q) = \mathcal{F}_{OSV}(p, \phi) - q_I \phi^I$
where $\mathcal{F}_{OSV}(p, \phi) = 4\text{Im}F(Y, \overline{Y})$
(we suppress $\Upsilon$ in the following) yields electric attractor equations:

$$\frac{\partial \mathcal{F}_{OSV}}{\partial \phi^I} = q_I$$

and black hole entropy = partial Legendre transform ($ReY^I \rightarrow q^I$) of free energy:

$$\mathrm{S}_{\mathrm{macro}}(p, q) = \pi \Sigma_* = \pi \left( \mathcal{F}_{OSV} - \phi^I \frac{\partial \mathcal{F}_{OSV}}{\partial \phi^I} \right)_*$$

# OSV conjecture (2)

Observation:

$$e^{\pi \mathcal{F}_{OSV}} = |e^{F_{\rm top}}|^2 = |Z_{\rm top}|^2$$

where $F_{\rm top} \propto iF(Y^I, \Upsilon)$ is the (properly normalized) all-genus free energy of the topological type-II string, and $Z_{\rm top}$ = all-genus topological partition function.

This is 'just' the relation between the topological string and couplings in the effective action.

Conjecture:

$$e^{\pi \mathcal{F}_{OSV}} = Z_{BH}(p, \phi)$$

should be interpreted as the partition function of the black hole,

$$Z_{BH}(p, \phi) = \sum_q d(p, q) e^{\pi q_I \phi^I}$$

$d(p, q)$ : microscopic degeneracy (microcanonical ensemble: electric and magnetic charges fixed).
$Z_{BH}(p, \phi)$ corresponds to a mixed ensemble: magnetic charges are fixed (microcanonical), electric charges fluctuate (canonical). Ensembles related by a (discrete) Laplace transform.

# OSV conjecture (3)

Inverse Laplace transform gives prediction of state degeneracy:

$$d(p,q) = \int_C d\phi^I \; e^{\pi \mathcal{F}_{OSV} - q_I \phi^I} \; .$$

Summary: OSV conjecture relates black hole partition function to partition function of the topological string

$$Z_{BH} = |Z_{\rm top}|^2 \; .$$

Weak version: this holds asymptotically in the semiclassical limit = limit of large charges. E.g. to all orders in $1/Q$, $Q =$ charges.
Successfully tested for ‘large’ black holes, not so clear for ‘small’ black holes.

Strong version: exact statement, once appropriate amendments are made (i.p. giving up holomorphic factorization). Intriguing, but much less clear.

Tests: (1) predict free energy $\mathcal{F}$ from microscopic state degeneracy $d(p,q)$, or vice versa. (2) Besides matching numbers (expansion coefficients), try to match structures in $Z_{BH}$ and $Z_{\rm top}$ (modular forms, Rademacher-type expansions). I.p. explain $Z_{BH} = |Z_{\rm top}|^2$.

# Open Questions Concerning OSV

- Convergence of sums/integrals, and choice of integration contours? Periodicity properties?
  This becomes relevant when going beyond a saddle point approximation.
- Electric-magnetic duality is not manifest. What about S- and T-duality invariance? Related point: what about nonholomorphic corrections?
  We will discuss a manifestly duality invariant version of the OSV conjecture below, and review the evidence for the presence of a non-trivial measure factor.
- Is $d(p, q)$ the absolute state degeneracy or an index (weighted sum)? Reminder: we need to vary parameters, and BPS states can (i) pair up into non-BPS states or (ii) decay when crossing lines of marginal stability. From the microscopic point of view, indices (like the elliptic genus) appear to be natural, but entropy is normally associated with absolute state degeneracy.

# Nonholomorphic Corrections

- The topological string has a holomorphic anomaly.
M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa (1993), E. Witten (1993),
Consequences for OSV have been discussed from this point of view by
R. Dijkgraaf, S. Gukov, A. Neitzke and C. Vafa (2005), E. Verlinde (2005), ...
- Effective field theory: Wilsonian effective action (local, IR cut-off) vs. generating functional of 1PI graphs (non-local i.g, when massless modes are present).
Physical, duality covariant couplings include non-holomorphic corrections.
L. Dixon, V. Kaplunovski and J. Louis (1991)
- The same applies to the entropy of BPS black holes. Generalization of attractor equations and variational principle:

$$\mathrm{Im}(F(Y,\Upsilon)) \rightarrow \mathrm{Im}(F(Y,\Upsilon)) + 2\Omega(Y,\overline{Y},\Upsilon,\overline{\Upsilon})$$

where $\Omega$ is real-valued, homogenous of degree 2 and (i.g.) not harmonic.
G.L. Cardoso, B. de Wit and T.M. (1999), G.L. Cardoso, B. de Wit, J. Käppeli and T.M. (2006).

# A modified version of OSV

Recall variational principle: Entropy = full Legendre transform of Hesse potential. This suggests:

$$
\begin{aligned}
e^{2\pi H(x,y)} &\approx Z_{\mathrm{BH}}^{(\mathrm{can})} = \sum_{p,q} d(p,q) e^{2\pi [q_I x^I - p^I y_I]} \\
\Leftrightarrow e^{\pi \mathcal{F}(Y,\overline{Y})} &\approx Z_{\mathrm{BH}}^{(\mathrm{can})} = \sum_{p,q} d(p,q) e^{\pi [q_I (Y^I + \overline{Y}^I) - p^I (F_I + \overline{F}_I)]}
\end{aligned}
$$

(we suppressed $\Upsilon$, but inclusion of $R^2$ and nonholomorphic corrections is understood.)

- Canonical rather than mixed ensemble. $(x, y) \propto$ electrostatic and magnetostatic potential.
- Symplectic covariance manifest.
- Nonholomorphic corrections are included ab initio.
- So far, we only have evidence for the weak form of the conjecture.

G.L. Cardoso, B. de Wit, J. Käppeli and T.M. (2006).

# A modified version of OSV (2)

Formal inverse Laplace transform gives state degeneracy in terms of black hole free energy:

$$d(p,q) \approx \int dx dy e^{\pi \Sigma(x,y)} \approx \int dY d\overline{Y} \Delta^-(Y,\overline{Y}) e^{\pi \Sigma(Y,\overline{Y})}$$

where

$$\Delta^{\pm}(Y,\overline{Y}) = |\det \left[ \mathrm{Im} F_{KL} + 2\mathrm{Re}(\Omega_{KL} \pm \Omega_{K\overline{L}}) \right] |$$

$\Delta^- =$ measure factor, $\Delta^+ =$ fluctuation determinant of saddle point integral.

Microscopic entropy:

$$d(p,q) = e^{\mathrm{S}_{\mathrm{micro}}(p,q)}$$

Macroscopic entropy, in saddle point approximation:

$$d(p,q) \approx e^{\mathrm{S}_{\mathrm{macro}}(p,q)} \sqrt{\frac{\Delta^-}{\Delta^+}} \approx e^{\mathrm{S}_{\mathrm{macro}}(p,q)(1+\cdots)}$$

Recall: $\mathrm{S}_{\mathrm{macro}}(p,q) = \pi \Sigma_*$. Note: $\Delta^{\pm}$ are subleading.

# Comparison to OSV

Partial saddle point approximation wrt Im$Y^I$ $\Leftrightarrow$ Imposing magnetic attractor equations. (Electric attractor equations follow from extremization of reduced entropy function.)

$$d(p,q) \approx \int d\phi \sqrt{\Delta^-(p,\phi)}\, e^{\pi[\mathcal{F}_E(\phi,p) - q_I \phi^I]}$$

$\mathcal{F}_E(\phi,p) =$ free energy in mixed ensemble. Includes nonholomorphic corrections encoded in $\Omega$!

Invert and compare to OSV:

$$\sqrt{\Delta^-}\, e^{\pi \mathcal{F}_E(p,\phi)} \approx Z_{\mathrm{BH}}^{(\mathrm{mix})} = \sum_q d(p,q) e^{\pi q_I \phi^I}$$

$$e^{\pi \mathcal{F}_{OSV}(p,\phi)} \approx Z_{\mathrm{BH}}^{(\mathrm{mix})} = \sum_q d(p,q) e^{\pi q_I \phi^I}$$

Original OSV conjecture does not have the measure factor $\Delta^-$ (which is implied by symplectic covariance) and does not include non-holomorphic corrections in the free energy. State counting shows that these additional terms are indeed present.

# $\mathcal{N}=4$ compactifications

Consider Het/$T^6$ ($\Leftrightarrow$ II/$(K3\times T^2)$).
Moduli space: $SL(2,\mathbb{R})/SO(2)\otimes SO(22,6)/(SO(22)\otimes SO(6))$.
Gauge group (for generic moduli:) $U(1)^{28}$.
Electric and magnetic charges: $(p,q)\in\Gamma_{22;6}\oplus\Gamma_{22;6}$.
T-duality group: $SO(22,6,\mathbb{Z})$.
S-duality group: $SL(2,\mathbb{Z})$

$\mathcal{N}=4$ orbifolds (CHL models) of order $N=2,3,5,7$.
Reduced rank of gauge group: $r=\frac{48}{N+1}+4=20,16,12,10$.
Reduced S-duality group $\Gamma_1(N)\subset SL(2,\mathbb{Z})$.

$$\Gamma_1(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{Z})|c=0 \text{ mod } N\ ,\quad a,d=1\text{ mod } \ N\right\}$$

# Counting $\frac{1}{2}$-BPS states

Consider Het/$T^6$. Basis for $\frac{1}{2}$-BPS states (light cone gauge):

$$\alpha^{i_1}_{-m_1}\alpha^{i_2}_{-m_2}\cdots|q\rangle \otimes |\mathbf{8}\oplus\mathbf{8}\rangle$$

$i_l = 1, \ldots, 24$ (transverse directions bosonic sector).
$m_k = 1, 2, \ldots$ excitation levels.
$q \in \Gamma_{22;6} =$ electric charges (Narain lattice).
$\mathbf{8}\oplus\mathbf{8} =$ ground state of supersymmetric sector.
A. Dabholkar and J. Harvey (1989)

Level matching:

$$n = \sum_k m_k \stackrel{!}{=} \frac{1}{2}|q^2| - 1$$

($q^2 < 0$ in our conventions.)
Excitation level fixed by the charges.

State counting problem $\Leftrightarrow$ Counting partitions of (large) integer $n$.

G.H. Hardy and S. Ramanujan (1918).

# Counting $\frac{1}{2}$-BPS states (2)

State counting problem ⇔ Counting partitions of (large) integer $n$.

G.H. Hardy and S. Ramanujan (1918).

Familiar from string partition functions.

$$d(q) = d(q^2) = 16 \oint d\tau \frac{\exp(i\pi\tau q^2)}{\eta^{24}(\tau)}$$

where $\tau = \tau_1 + i\tau_2 \in \mathcal{H}$ (upper half plane) and $\eta(\tau) =$ Dedekind $\eta$-function.
$\eta^{24}$ is a modular form of weight 12.

Systematic method of asymptotic expansion: Rademacher expansion, aka. Farey tail expansion.

H. Rademacher (1938), see R. Dijkgraaf, J. Maldacena, G. Moore and E. Verlinde (2000).

# Counting $\frac{1}{2}$-BPS states (3)

Result for $\frac{1}{2}$-BPS states of heterotic string:

$$d(q^2) = 16 \sum_{c=1}^{\infty} c^{-14} \mathsf{Kl}(\tfrac{1}{2}|q^2|, -1; c) \hat{I}_{13}\left(\frac{4\pi}{c}\sqrt{\frac{1}{2}|q^2|}\right)$$

$\hat{I}_{13} =$ modified Bessel functions, Kl= 'Kloosterman sums'. Contributions $c > 1$ are exponentially suppressed for large $|q^2|$. Leading term

$$d(q^2) = 16\ \hat{I}_{13}(4\pi\sqrt{\tfrac{1}{2}|q^2|})$$

can be further expanded:

$$\mathrm{S}_{\mathrm{micro}}(q^2) = \log d(q^2) \approx 4\pi\sqrt{\tfrac{1}{2}|q^2|} - \frac{27}{4}\log|q^2| + \frac{15}{2}\log(2) - \frac{675}{32\pi|q^2|} + \cdots$$

(Taken from `A. Dabholkar, F. Denef, G. Moore and B. Pioline (2005)`)

First two terms = Cardy formula = saddle point evaluation of the integral representation. (First term = value of integrand at saddle point, second term = fluctuation determinant.)

# Remark

The Rademacher expansion can be generalized to Jacobi modular forms (elliptic genera). Using this and $AdS^3/CFT_2$, the (asymptotic) holomorphic factorization

$$Z_{\rm BH} \approx |Z_{\rm top}|^2$$

was explained for certain black holes in $\mathcal{N}=2$ compactifications. The factorization is due to contributions from both branes and antibranes ($M2/\overline{M2}$ or $D2/\overline{D2}$).

D. Gaiotto, A. Strominger and X. Yin, hep-th/0602046, P. Kraus and F. Larsen, hep-th/0607138, C. Beasley, D. Gaiotto, M. Guica, L. Huang, A. Stominger and X. Yin, hep-th/0608021, J. de Boer, M.C.N. Cheng, R. Dijkgraaf, J. Manshot and E. Verlinde, hep-th/0608059.

# Counting $\frac{1}{4}$-BPS states

$\frac{1}{4}$-BPS states of Het/$T^6$: bound states of fundamental strings (electric) and heterotic five-branes (magnetic).

State degeneracy:

$$d(p,q) = \oint d\rho d\sigma dv \frac{e^{i\pi[\rho p^2 + \sigma q^2 + (2v-1)pq)]}}{\Phi_{10}(\rho,\sigma,v)}$$

Contour integral in the Siegel upper half space:

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}$$ complex, symmetric, positive definite imaginary part.

$\Phi_{10} =$ weight 10 Siegel cusp form (generalizes $\eta^{24}$).

(Strong) motivation for the above formula:

- Dual type-II picture: worldvolume theory of NS5 brane = string field theory (free limit sufficient).
  `R. Dijkgraaf, E. Verlinde and H. Verlinde (1996)`
- Using the D1-D5 system: `S. Shih, A. Strominger and X. Yin (2005)`.

# Counting $\frac{1}{4}$-BPS states (2)

Extension to CHL models: $\Phi_{10}$ is replaced by a cusp form $\Phi_k$ of weight $k$, where $(k+2)(N+1) = 24$.

$$d(p,q) = \oint d\rho d\sigma dv \frac{e^{i\pi[\rho p^2 + \sigma q^2 + (2v-1)pq)]}}{\Phi_k(\rho, \sigma, v)}$$

`D.P. Jatkar and A. Sen (2005)`

To test the predictions of OSV, we also need $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS black hole solutions of $\mathcal{N} = 4$ supergravity, including subleading terms.

# $\mathcal{N}=4$ Supergravity in $\mathcal{N}=2$ formalism

The relevant subsector of the $\mathcal{N}=4$ theory is described by $\mathcal{N}=2$ vector multiplets with prepotential

$$F(Y,\Upsilon) = -\frac{Y^1 Y^a \eta_{ab} Y^b}{Y^0} + F^{(1)}(S)\Upsilon$$

where $S = -i\frac{Y^1}{Y^0}$ is the heterotic dilaton, $a = 2,\ldots,n$.
Note $F^{(g>2)} = 0$ and $F^{(1)} = F^{(1)}(S)$.

Dilaton $S$ is T-duality invariant and transforms under S-duality as:

$$S \to \frac{aS - ib}{icS + d}$$

Scalar products $(q^2, p^2, p\cdot q)$ are T-duality invariant and transform in the $\mathbf{3}$-representation of $SL(2,\mathbb{Z})_S$.

# Reduced variational principle

All attractor equations can be solved except those which determine the dilaton $S$. The dilaton attractor equations determine the critical points of the reduced entropy function

$$\Sigma(S,\overline{S},p,q) = -\frac{q^2 - ip\cdot q(S-\overline{S}) + p^2|S|^2}{S+\overline{S}} + 4\Omega(S,\overline{S},\Upsilon,\overline{\Upsilon})$$

(We absorbed $F^{(1)}(S)$ into $\Omega(S,\overline{S},\Upsilon,\overline{\Upsilon})$.)

Dilaton attractor equations:

$$\partial_S\Sigma = 0 = \partial_{\overline{S}}\Sigma$$

Entropy

$$\mathrm{S}_{\mathrm{macro}}(p,q) = \pi\Sigma_*$$

is manifestly T- and S-duality invariant if $\Omega$ is S-duality invariant.

`G.L. Cardoso, B.de Wit and T.M. (1999), G.L. Cardoso, B.de Wit, J. Käppeli and T.M. (2006).`

# $\mathcal{N} = 4$ BPS black holes

At the two-derivative level, BPS black hole entropy is:

$$\mathrm{S}_{\mathrm{macro}} = \pi\sqrt{q^2p^2 - (q \cdot p)^2}$$

M. Cvetic and D. Youm, (1995), E. Bergshoeff, R. Kallosh and T. Ortin (1996).

Two cases:

1. Dyonic $\frac{1}{4}$-BPS black holes with non-vanishing area, 'large black holes.'
   $q^2p^2 - (q \cdot p)^2 \neq 0 \Leftrightarrow M = |Z_1| > |Z_2|$.
2. 'Electric' $\frac{1}{2}$-BPS black holes with vanishing area (at leading order), 'small black holes.'
   $q^2p^2 - (q \cdot p)^2 = 0 \Leftrightarrow M = |Z_1| = |Z_2|$.

# Large black holes

Evaluation of

$$d(p,q) = \oint d\rho d\sigma dv \frac{e^{i\pi[\rho p^2 + \sigma q^2 + (2v-1)pq)]}}{\Phi_{10}(\rho,\sigma,v)}$$

at leading order (saddle point evaluation without fluctuation determinant) gives

$$\mathrm{S}_{\mathrm{micro}} = \log d(p,q) \approx \pi\sqrt{q^2p^2 - (q\cdot p)^2} \approx \mathrm{S}_{\mathrm{macro}}$$

`R. Dijkgraaf, E. Verlinde and H. Verlinde (1996)`

This extends to CHL models.

# $R^2$ and nonholomorphic corrections

$R^2$ corrections and nonholomorphic corrections are encoded in

$$\Omega = \frac{1}{256\pi}\left[\Upsilon \log\eta^{24}(S) + \overline{\Upsilon}\log\eta^{24}(\overline{S}) + \tfrac{1}{2}(\Upsilon+\overline{\Upsilon})\log(S+\overline{S})^{12}\right]$$

J.A. Harvey and G.W. Moore (1996)

Note:

$$\log\eta^{24}(S) = -2\pi S - e^{-2\pi S} + \mathcal{O}(e^{-4\pi S})$$

Infinite series of space-time instanton corrections.

Saddle point evaluation (including fluctuation determinant) of

$$d(p,q) = \oint d\rho d\sigma dv \frac{e^{i\pi[\rho p^2+\sigma q^2+(2v-1)pq)]}}{\Phi_{10}(\rho,\sigma,v)}$$

gives $\mathrm{S}_{\mathrm{micro}}(p,q) \approx \pi\Sigma_* \approx \mathrm{S}_{\mathrm{macro}}(p,q)$.

Saddle point equations = dilaton attractor equations.

Semiclassical result, includes the non-perturbative terms in $\Omega$.

G.L. Cardoso, B. de Wit, J. Käppeli and T.M. (2004)

# OSV for large black holes

Evaluate mixed partition function $Z(p,\phi) = \sum_q d(p,q) e^{\pi q_I \phi^I}$ using integral representation of $d(p,q)$. Result:

$$Z(p,\phi) = \sum_{\text{shifts}} \sqrt{\tilde{\Delta}(p,\phi)}\, e^{\pi \mathcal{F}_E(p,\phi)}$$

(sum over shifts to enforce periodicity), where the free energy

$$\mathcal{F}_E(p,\phi) = \tfrac{1}{2}(S+\overline{S})\left(p^a \eta_{ab} p^b - \phi^a \eta_{ab} \phi^b\right) - i(S-\overline{S}) p^a \eta_{ab} \phi^b + 4\Omega(S,\overline{S},\Upsilon,\overline{\Upsilon})$$

includes $R^2$ and nonholomorphic corrections. Measure factor asymptotically agrees with prediction from modified OSV conjecture:

$$\tilde{\Delta}^- \approx \Delta^-$$

(Holds for CHL models as well)

`D. Shih and X. Yin (2005), G.L. Cardoso, B. de Wit, J. Käppeli and T.M. (2006)`

# OSV for large black holes (2)

- Highly non-trivial test of OSV conjecture.
- Restricted to semi-classical limit, no statement about strong version of the conjecture.
- Presence of non-trivial measure factor established.
- Observation: the number of $\mathcal{N} = 2$ vector multiplets needed for matching is 27 (thus accounting for all 28 = 1 + 27 gauge fields), rather than 23 (corresponding to a naive $\mathcal{N} = 2$ truncation). (Same shift for CHL models.)

# OSV for small black holes

Solution based on two-derivative effective action:

$$\mathrm{S}_{\mathrm{macro}} = \pi\sqrt{p^2q^2 - (p\cdot q)^2} = 0 \quad \text{for} \quad p = 0$$

Area: $A = 0$, null singularity.
Scalars attracted to the boundary of moduli space, i.p. dilaton $S = \infty$.

Entropy disagrees with leading order string state counting:

$$\mathrm{S}_{\mathrm{micro}} \approx 4\pi\sqrt{\tfrac{1}{2}|q^2|}$$

# OSV for small black holes (2)

First subleading correction is the (heterotic) tree level $R^2$-term encoded in

$$\log \eta^{24}(S) = -2\pi S + \mathcal{O}(e^{-2\pi S})$$

Stringy cloaking of the null singularity:

$$A = 8\pi\sqrt{\tfrac{1}{2}|q^2|} \neq 0$$

$R^2$-corrections generate a finite horizon.

Entropy

$$\mathrm{S}_{\mathrm{macro}} = \frac{A}{4} + \text{Wald's correction} = \frac{A}{4} + \frac{A}{4} = \frac{A}{2} = 4\pi\sqrt{\tfrac{1}{2}|q^2|}$$

agrees with leading order $\mathrm{S}_{\mathrm{micro}}$. Wald's modification of the area law is crucial.
`A. Dabholkar, R. Kallosh and A. Maloney (2004).`

What about subleading terms in the entropy?

# OSV for small black holes (3)

Including the non-holomorphic corrections:

$$\begin{aligned} \mathrm{S}_{\mathrm{macro}}^{(\mathrm{Wald})} &= 4\pi\sqrt{\tfrac{1}{2}|q^2|} - 6\log|q^2| + \cdots \\ \mathrm{S}_{\mathrm{micro}}^{(Cardy)} &= 4\pi\sqrt{\tfrac{1}{2}|q^2|} - \frac{27}{4}\log|q^2| + \cdots \end{aligned}$$

However, both entropies refer to different ensembles, according to OSV. Our modified version of the conjecture implies

$$\mathrm{S}_{\mathrm{micro}} = \mathrm{S}_{\mathrm{macro}} + \log\sqrt{\frac{\Delta^-}{\Delta^+}}$$

But for electric black holes

$$\begin{aligned} \Delta^- &= 0 \quad \text{up to non-holomorphic terms and instantons} \\ \Delta^+ &= 0 \quad \text{up to instantons} \end{aligned}$$

# OSV for small black holes (3)

Cannot perform saddle point approximation of the full integral, because leading order solution is singular.

We can still test the idea that OSV has to be modified by our measure factor $\Delta^-$ and by nonholomorphic terms by evaluating

$$\exp(\mathrm{S}_{\mathrm{micro}}) = d(p^1, q) \approx \int d\phi \sqrt{\Delta^-(p^1, \phi)} e^{\pi[\mathcal{F}_E(p^1, \phi) - q_I \phi^I]}$$

when including the nonholomorphic terms in $\Delta^-$.
Note: $p^1$ is an electric charge (for the heterotic string).

Neglecting instanton corrections, we find:

$$d(p^1, q) \approx \int \frac{dS d\overline{S}}{(S + \overline{S})^{k+4}} \sqrt{S + \overline{S} - \frac{k+2}{2\pi}} \exp\left[ -\frac{\pi q^2}{S + \overline{S}} + 2\pi(S + \overline{S}) \right]$$

where $k = 10$ for Het/$T^6$ and other values for CHL models.

# OSV for small black holes (4)

Thus (for $k = 10$)

$$\begin{aligned} S^{(\mathrm{mod.OSV})}_{\mathrm{predicted}} &\approx \hat{I}_{13-\frac{1}{2}}(4\pi\sqrt{\tfrac{1}{2}|q^2|}) \approx 4\pi\sqrt{\tfrac{1}{2}|q^2|} - \frac{13}{2}\log|q^2| + \cdots \\ S_{\mathrm{micro}} &\approx \hat{I}_{13}(4\pi\sqrt{\tfrac{1}{2}|q^2|}) \approx 4\pi\sqrt{\tfrac{1}{2}|q^2|} - \frac{27}{4}\log|q^2| + \cdots \end{aligned}$$

G.L. Cardoso, B. de Wit, J. Käppeli and T.M., hep-th/0601108.
Slight but systematic mismatch of subleading $\log$ and inverse power corrections. Same for CHL models. Shift is due to the measure. Maybe there is no measure?

Unmodified OSV conjecture: no measure, no nonholomorphic terms:

$$d(p^1, q) \approx \int d\phi e^{\pi[\mathcal{F}_{OSV}(p^1,\phi) - q_I\phi^I]} \approx (p^1)^2 \hat{I}_{13}(4\pi\sqrt{\tfrac{1}{2}|q|})$$

Factor $(p^1)^2$ spoils T-duality. (Some) Measure needed.
Index $\nu = 13$ of Bessel function requires to truncate (unmodified) OSV integral to 24 potentials. The dyonic case works with 28.

A. Dabholkar, F. Denef, G. Moore and B. Pioline (2005)

# OSV for small black holes (5)

- For small black holes, there are discrepancies at semiclassical limit beyond the leading term.
- A measure factor seems to be needed for duality invariance.
- Problems are related to singular behaviour of ‘leading order’ solution. How to set up a well defined expansion?
- When including instanton corrections, the structure of OSV-type integrals looks different from integral representations of state degeneracies. Unclear how strong version of the conjecture could work.

# OSV for small black holes (6)

One more result from the comprehensive study of small $\mathcal{N}=4$ and $\mathcal{N}=2$ black holes performed by `A. Dabholkar, F. Denef, G. Moore and B. Pioline (2005)`. (See also nice review by `B. Pioline, hep-th/0607227`)

- In twisted sectors of $\mathcal{N}=2$ orbifolds, absolute and indexed degeneracies are equal and agree with OSV.
- In the untwisted sector of $\mathcal{N}=2$ orbifolds, the leading order absolute degeneracies agree with OSV. Indexed degeneracies are exponentially smaller.

# Concluding remarks

- OSV story far from complete.
- We didn't discuss implications of OSV for flux compactifications, cosmology, etc.
- A full understanding of black hole entropy includes characterizing black hole microstates in the gravitational regime. Concrete proposal: one-one correspondence between BPS states and smooth BPS space-times. Reviewed in `S.D. Mathur, hep-th/0510180`.

  Is there a relation to the interpretation of the Rademacher/Farey tail expansion in terms of a supergravity partition function?

  `R. Dijkgraaf, J. Maldacena, G. Moore and E. Verlinde (2000), P. Kraus and F. Larsen, hep-th/0607138, J. de Boer, M.C.N. Cheng, R. Dijkgraaf, J. Manshot and E. Verlinde, hep-th/0608059.`

# Appendix1: BPS solitons (long)

BPS states can be realized as finite energy solutions $\Phi_0$ of the field equations (asymptotic to vacuum).

Killing spinors $\varepsilon \leftrightarrow$ residual (rigid) supersymmetry of $\Phi_0$:

$$\delta_\varepsilon \, \Phi|_{\Phi_0} = 0 \, .$$

Examples:

- BPS monopoles and dyons in Super-Yang-Mills theories.
- Extremal black hole and black p-brane solutions of Supergravity theories.
- Supersymmetric compactifications of Supergravity theories.

# Extremal Reissner-Nordstrom solution

Extremal Reissner-Nordstrom solution of Einstein-Maxwell theory:

$$ds^2 = -H^{-2}(r)dt^2 + H^2(r)d\vec{x}^2 \ , \quad F_{tr} = \partial_r H^{-1}(r) \ ,$$

where $H = 1 + \frac{q}{r}$ is harmonic, and $q = M > 0$ electric charge.

# Extremal Reissner-Nordstrom solution

Extremal Reissner-Nordstrom solution of Einstein-Maxwell theory:

$$ds^2 = -H^{-2}(r)dt^2 + H^2(r)d\vec{x}^2 \ , \quad F_{tr} = \partial_r H^{-1}(r) \ ,$$

where $H = 1 + \frac{q}{r}$ is harmonic, and $q = M > 0$ electric charge.

Special properties:

- Saturation of mass bound for existence of horizon: $M \geq |q|$.
- Vanishing Hawking temperature: $T \sim \kappa_S = 0$.
- Near horizon limit is maximally symmetric (Bertotti-Robinsion solution):

$$ds^2 = -\frac{r^2}{q^2}dt^2 + \frac{q^2}{r^2}dr^2 + q^2 d\Omega^2 \quad (AdS^2 \times S^2) \ ,$$

$$\nabla_\mu F_{\nu\rho} = 0 \rightarrow F_{tr} = \text{const} \sim q \ , \quad \text{constant field strength = flux .}$$

- Static multicentered solutions: $H(r) \rightarrow H = 1 + \sum_{i=1}^{N} \frac{q_i}{|\vec{x} - \vec{x}_i|}$.

# Extremal RN = Supersymmetric Soliton

G. Gibbons (1982)

Embedd Einstein-Maxwell theory into $\mathcal{N} = 2$ Supergravity $\{e_\mu{}^a, \psi^i_\mu, A_\mu\}$.
Graviphoton $A_\mu$ gauges central charge,

$$Z = q + ip$$

$q =$ electric charge, $p =$ magnetic charge.

- Extremal RN black hole is $\frac{1}{2}$-BPS solution: 4 Killing spinors.
- Black hole mass bound = supersymmetric mass bound: $M^2 \geq |Z|^2 = q^2 + p^2$.
- Extremal RN black hole interpolates between two maximally supersymmetric solutions (supersymmetric vacua):

| | | | | |
|---|---|---|---|---|
| Minkowski space | $\overset{r\to\infty}{\longleftarrow}$ | Extr. RN | $\overset{r\to 0}{\longrightarrow}$ | $AdS^2 \times S^2$ |
| 8 Killing spinors | | 4 Killing spinors | | 8 Killing spinors |

- Multicentered solution: cancellation of gravitational attraction and electrostatic repulsion through (super)symmetry.

# Electric–Magnetic Duality

What about magnetic charges?

Field equations

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu {}^*F_{\rho\sigma} = 0 \quad \Leftrightarrow \quad \nabla^\mu F_{\mu\nu} = 0$$
$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$$

invariant under $Sp(2,\mathbb{R}) \simeq SL(2,\mathbb{R})$ rotations of the field strength $(F_{\mu\nu}, {}^*F_{\mu\nu})^T$.

Electric – magnetic duality.

Action on solutions: rotate charges $(q,p)^T$, generate dyonic solutions out of electric ones:

$$M = q = |Z| \longrightarrow M = \sqrt{q^2 + p^2} = |Z| \; .$$

# Appendix2: The Rademacher expansion

Ingredients:

1. A modular form $p(\tau)$ of positive weight $w > 0$ can be reconstructed from its singular part $p^{-}(\tau)$ (the negative powers of $e^{2\pi i\tau}$ in its Laurent series) by the Poincaré series:

$$p(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} p^{-}(\gamma\tau)(c\tau + d)^{-w}$$

   where $\Gamma = SL(2, \mathbb{Z})$ and $\Gamma_\infty =$ stabilizer of the cusp point $i\infty$ (translations).

2. One would like to apply this to

$$\eta^{-24}(\tau) = \frac{1}{e^{2\pi i\tau}} + 24 + \mathcal{O}(e^{2\pi i\tau}) \;, \quad \text{for} \quad \tau \rightarrow i\infty \;,$$

   which, however, has negative weight. Apply Poincaré series to the ‘Farey tail transform’:

$$\hat{p}(\tau) = \left( e^{2\pi i\tau} \frac{d}{d(e^{2\pi i\tau})} \right)^{1-w} p(\tau)$$

   which has positive weight.

# The Rademacher expansion (2)

Result for $\frac{1}{2}$-BPS states of heterotic string:

$$d(q^2) = 16 \sum_{c=1}^{\infty} c^{-14} \mathsf{Kl}(\tfrac{1}{2}|q^2|, -1; c) \hat{I}_{13} \left( \frac{4\pi}{c} \sqrt{\frac{1}{2}|q^2|} \right)$$

$\hat{I}_{13} =$ modified Bessel functions, Kl= 'Kloosterman sums'. Contributions $c > 1$ are exponentially suppressed for large $|q^2|$. Leading term

$$d(q^2) = 16 \, \hat{I}_{13}(4\pi \sqrt{\tfrac{1}{2}|q^2|})$$

can be further expanded:

$$\mathrm{S}_{\mathrm{micro}}(q^2) = \log d(q^2) \approx 4\pi \sqrt{\tfrac{1}{2}|q^2|} - \frac{27}{4} \log |q^2| + \frac{15}{2} \log(2) - \frac{675}{32\pi |q^2|} + \cdots$$

(Taken from `A. Dabholkar, F. Denef, G. Moore and B. Pioline (2005)`)

First two terms = Cardy formula = saddle point evaluation of the integral representation. (First term = value of integrand at saddle point, second term = fluctuation determinant.)

# The Rademacher expansion (3)

Generalizations:

- For $\mathcal{N} = 4$ orbifolds, $\eta^{24}$ is replaced by the cusp form $f^{(k)}$ of $\Gamma_1(N)$, which has weight $k + 2 = \frac{24}{N+1}$.
- In $\mathcal{N} = 2$ compactifications states are counted by (some version of) the elliptic genus,

$$\chi(\tau, z) = \text{Tr}_{RR}\left((-1)^F e^{2\pi i\tau L_0 - 2\pi i\overline{\tau}\overline{L}_0} e^{2\pi i z J_0 - 2\pi i\overline{z}\overline{J}_0}\right) \; , \quad \text{(schematically)}$$

which is a Jacobi modular form ($z$ transforms like a worldsheet coordinate). Generalizations of the Rademacher formula involve vector-valued modular forms:

$$\chi(\tau, z) = \sum_{\mu} \chi_\mu(\tau)\theta_\mu(\tau, z)$$

$\mu = SL(2, \mathbb{Z})$ representation.

`J. de Boer, M.C.N. Cheng, R. Dijkgraaf, J. Manshot and E. Verlinde (2006)`

# The Rademacher expansion (4)

Interpretation of the Rademacher expansion/Farey tail expansion in terms of a supergravity partition function:

- Poincaré series $\sum_{\Gamma_\infty \backslash \Gamma}$ = Sum over saddle points. In the $AdS^3/CFT_2$ set-up saddle points are black holes.
- Remaining sum over pole terms = Sum over excitations below the black hole formation threshold.

J. de Boer, M.C.N. Cheng, R. Dijkgraaf, J. Manshot and E. Verlinde (2006)

The excitations correspond to a dilute gas of $M2$ and $\overline{M2}$ branes, equivalently of $D2$ and $\overline{D2}$ branes, in $AdS^{2/3} \times S^2$. This implies (asymptotically) the holomorphic factorization

$$Z_{\rm BH} \approx |Z_{\rm top}|^2 \ .$$